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Time reversal and reflected diffusions

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Abstract

We extend Föllmer's results on time reversal on Wiener space to the case of some reflected diffusions. © 1997 Elsevier Science B.V.

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1. Introduction

Let us consider the following problem. Let A be the generator of a $\mathbb{R}^n$-valued diffusion process, β a vector field defined on $\mathbb{R}^n$, and $(\rho_t(\mathrm{d}x))_{0\leqslant t\leqslant 1}$ a weakly continuous flow of probability measures defined on $\mathbb{R}^n$, such that the following weak Fokker–Planck equation is satisfied (i.e. in the sense of Schwartz distributions):

$$\frac{\partial}{\partial t}\rho_t = (A+\beta\nabla)^*\rho_t. \tag{1.1}$$

We may then ask natural questions like the existence of a diffusion process with generator $(A+\beta\nabla)$ and marginals $(\rho_t(\mathrm{d}x))_{0\leqslant t\leqslant 1}$; then, if it exists, how can we describe the reversed process, and recover the drifts? When $A=\frac{1}{2}\triangle$ and $\rho_t(\mathrm{d}x)=\rho_t(x)\,\mathrm{d}x$, this problem comes from Nelson's stochastic mechanics (Nelson, 1967, 1988). Carlen (1984) studied it first, using a semi-group approach, and some more assumptions, like

$$\int_0^1\int|\beta(t,x)|^2\,\rho(t,x)\,\mathrm{d}x\,\mathrm{d}t<+\infty \quad \text{finite energy condition,} \tag{1.2a}$$

$$\int_0^1\int|\hat\beta(t,x)|^2\,\rho(1-t,x)\,\mathrm{d}x\,\mathrm{d}t<+\infty \quad \text{dual finite energy condition.} \tag{1.2b}$$

Föllmer (1984, 1988) used another approach to the problem. In finite energy conditions (1.2), he recognized some entropic condition, as we now briefly recall. He

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considers a probability measure $\mathbb{Q}$ on $\mathscr{C}[0,1]$, which has finite relative entropy with respect to the Wiener measure $\mathbb{P}$:

$$H(\mathbb{Q}|\mathbb{P}) = \mathbb{E}\left[\operatorname{Log}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] < +\infty. \tag{1.3}$$

Thanks to Girsanov transformation theory, the canonical coordinate process $(x_t)_{0\leqslant t\leqslant 1}$ satisfies the stochastic differential equation

$$\mathrm{d}x_t = \mathrm{d}w_t + \beta_t\,\mathrm{d}t \tag{1.4}$$

where $(w_t)_{0\leqslant t\leqslant 1}$ is a Brownian motion under $\mathbb{Q}$, and $(\beta_t)_{0\leqslant t\leqslant 1}$ an adapted process with finite energy

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \beta_t^2\,\mathrm{d}t\right] < \infty. \tag{1.5}$$

Let R be the time reversal operator on $\mathscr{C}[0,1]$, i.e.

$$R(x) : (t \mapsto x_{1-t}). \tag{1.6}$$

Considering the reverse probability measures $\overline{\mathbb{P}} = \mathbb{P}\circ R$ and $\overline{\mathbb{Q}} = \mathbb{Q}\circ R$, Föllmer shows the coordinate process is also solution of a stochastic differential equation of type (1.4) with some adapted process $(\overline{\beta}_t)_{0\leqslant t\leqslant 1}$. Furthermore, with no regular assumption of the drifts, but only the finite energy condition (1.5), he shows the distribution of x_t under $\mathbb{Q}$ is absolutely continuous with some density function $\rho_t(x)$, and he gives a rigorous proof of the duality equation:

$$\mathbb{E}[\overline{\beta}_{1-t}\circ R + \beta_t / x_t = x] = (\operatorname{Log}\rho_t(x))'. \tag{1.7}$$

He also describes the drifts as forward and backward derivatives of the coordinate process.

Some generalizations of this problem are made in Petit (1992) and Cattiaux and Petit (1995) where $\mathbb{P}$ is the law of a general diffusion. Let us mention some more authors who studied analogous problems, like Fukushima and Takeda (1984), Carmona (1985), Meyer and Zheng (1985), Picard (1985), Zheng (1985) and Pardoux and Williams (1994). Nevertheless, the existence problem is never considered in all these papers. It has only been considered by Cattiaux and Léonard (1994, 1995a, b).

The aim of this paper is to extend Föllmer's approach to the case where $\mathbb{P}$ is no longer Wiener measure but the law of a reflected diffusion. To make clear the differences with the non-reflected case, in the first part of the paper, we study the problem when $\mathbb{P}$ is the law of a reflected Brownian motion, and we point out specific difficulties due to the apparition of local time. We establish the equation satisfied by the reversed process, we prove the duality equation, and show how to recover the drifts as forward and backward derivatives. The next section is devoted to the case where $\mathbb{P}$ is the law of a reflected diffusion $X_t = (Z_t, Y_t)$, the first component of which is a reflected Brownian motion Z_t with local time $l_t(Z)$; $\mathbb{P}$ is then the law of the solution of the stochastic differential equation:

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t + \tfrac{1}{2}\,\mu(Y_t)\,\mathrm{d}l_t(Z) \tag{1.8}$$

where

$$b(s,x)=\begin{pmatrix}0\\ \chi_0(s,x)\end{pmatrix},\qquad \sigma(x)=\begin{pmatrix}1 & 0\\ 0 & \chi_1(x)\end{pmatrix},\quad \text{and}\quad \mu(y)=\begin{pmatrix}1\\ V_0(y)\end{pmatrix}$$

are $\mathscr{C}_b^\infty$-smooth coefficients, and χ_1 is an invertible matrix field.

Notations. We denote by $l_t(a,U)$ the local time at time t and level a of the process (U_t); when $a=0$, we only write $l_t(U)$ to simplify.

$(\mathscr{F}_t^U)_{0\leqslant t\leqslant 1}$ is the filtration generated by the process $(U_t)_{0\leqslant t\leqslant 1}$, i.e. $\mathscr{F}_t^U=\sigma(U_s; s\leqslant t)$.

$\mathscr{C}_K^\infty(\mathbb{R}_+^*)$ (resp. $\mathscr{C}_K^\infty(\mathbb{R}_+^*\times\mathbb{R}^n)$) denotes the set of $\mathscr{C}^\infty$-smooth functions, $\mathbb{R}$-valued, defined on $[0,+\infty[$ (resp. $[0,+\infty[\times\mathbb{R}^n$), with compact support included in $]0,+\infty[$ (resp. $]0,+\infty[\times\mathbb{R}^n$).

2. The reflected Brownian motion case

Let $(Z_t)_{0\leqslant t\leqslant 1}$, be a reflected Brownian motion, starting at $Z_0\geqslant 0$, and solution of the stochastic differential equation

$$Z_t=|B_t-Z_0|=Z_0+W_t+l_t(Z_0,B) \tag{2.1}$$

where $(B_t)_{0\leqslant t\leqslant 1}$ and the process $(W_t\stackrel{\text{def}}{=}\int_0^t \operatorname{sgn}(B_s-Z_0)\,\mathrm{d}B_s)_{0\leqslant t\leqslant 1}$ are standard Brownian motions under $\mathbb{P}^*$. Under $\mathbb{P}^*$, the law of Z_t has a density with respect to Lebesgue measure:

$$p(t,z)=\frac{1}{\sqrt{2\pi t}}\left(\exp-\frac{(z-z_0)^2}{2t}+\exp-\frac{(z+z_0)^2}{2t}\right)1_{(z\geqslant 0)} \tag{2.2}$$

when Z starts from the constant z_0; and when the initial law of Z is p_0 (for technical reasons, we suppose the support of p_0 is compact, included in $[0,K]$),

$$p(t,z)=1_{(z\geqslant 0)}\int_{\mathbb{R}_+}\mathrm{d}p_0(z_0)(h(t,z-z_0)+h(t,z+z_0)) \tag{2.3}$$

where

$$h(t,z)=\frac{1}{\sqrt{2\pi t}}\exp-\frac{z^2}{2t}. \tag{2.4}$$

Then, we have

$$l_t(Z)=2\,l_t(Z_0,B). \tag{2.5}$$

So, Eq. (2.1) is equivalent to

$$Z_t=Z_0+W_t+\tfrac{1}{2}\,l_t(Z). \tag{2.6}$$

Furthermore, we know the filtrations generated by Z and W are such that $\mathscr{F}_t^W\subseteq\mathscr{F}_t^Z\subseteq\mathscr{F}_t^W\vee\sigma(Z_0)$. In fact, the equality $W_t=Z_t-Z_0-\frac{1}{2}l_t(Z)$ implies the first inclusion $\mathscr{F}_t^W\subseteq\mathscr{F}_t^Z$, while $Z_t=Z_0+W_t+\sup_{s\leqslant t}(-(W_s+Z_0)\vee 0)$ implies the second one $\mathscr{F}_t^Z\subseteq\mathscr{F}_t^W\vee\sigma(Z_0)$. In particular, $(W_t)_{0\leqslant t\leqslant 1}$ is a $\mathscr{F}_t^Z$-martingale.

Then, let us denote $\mathbb{P}=\mathbb{P}^*\circ Z^{-1}$, the law of Z under $\mathbb{P}^*$.

2.1. Equation satisfied by the reversed process

Let us denote $(\bar{Z}_t \overset{\text{def}}{=} Z_{1-t})_{0\leqslant t\leqslant 1}$ the reversed process of $(Z_t)_{0\leqslant t\leqslant 1}$.

Lemma 2.1. *Let $(\overline{W}_t)_{0\leqslant t\leqslant 1}$ be the real process defined by*

$$\overline{W}_t \overset{\text{def}}{=} \bar{Z}_t - \bar{Z}_0 - \frac{1}{2}l_t(\bar{Z}) - \int_0^t \frac{\frac{\partial}{\partial z}p(1-s,\bar{Z}_s)}{p(1-s,\bar{Z}_s)}\,\mathrm{d}s$$

$$= -W_1 + W_{1-t} - l_1(Z) + l_{1-t}(Z) - \int_{1-t}^1 \frac{\frac{\partial}{\partial z}p(s,Z_s)}{p(s,Z_s)}\,\mathrm{d}s$$

where $l_t(\bar{Z})$ denotes the local time at level 0 and time t of the reversed process $\bar{Z}$, i.e.

$$l_t(\bar{Z}) \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{(0\leqslant \bar{Z}_s\leqslant\varepsilon)}\,\mathrm{d}s = l_1(Z) - l_{1-t}(Z) = 2(l_1(Z_0,B) - l_{1-t}(Z_0,B)).$$

Then, the process $(\overline{W}_t)_{0\leqslant t\leqslant 1}$ is, under $\mathbb{P}^*$, a $\mathscr{F}^{\bar{Z}}$ Brownian motion, and the reversed process $\bar{Z}$ is solution of the stochastic differential equation

$$\bar{Z}_t = \bar{Z}_0 + \overline{W}_t + \frac{1}{2}l_t(\bar{Z}) + \int_0^t \frac{\frac{\partial}{\partial z}p(1-s,\bar{Z}_s)}{p(1-s,\bar{Z}_s)}\,\mathrm{d}s. \tag{2.7}$$

Proof. By definition, $(\overline{W}_t)_{0\leqslant t\leqslant 1}$ is $\mathscr{F}_t^{\bar{Z}}$ adapted, with bracket $\langle \overline{W}\rangle_t = t$, and $(\bar{Z}_t)_{0\leqslant t\leqslant 1}$ satisfies Eq. (2.7). Actually, the Markov property of Z implies, for any $0\leqslant s\leqslant t\leqslant 1$:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^*}[\overline{W}_t - \overline{W}_s/\sigma(\bar{Z}_u; u\leqslant s)]\\
&\quad = \mathbb{E}^{\mathbb{P}^*}\Big[-Z_{1-s} + Z_{1-t} - \tfrac{1}{2}(l_{1-s}(Z) - l_{1-t}(Z))\\
&\qquad - \int_{1-t}^{1-s}\frac{\frac{\partial}{\partial z}p(u,Z_u)}{p(u,Z_u)}\mathrm{d}u/\sigma(Z_u; u\geqslant 1-s)\Big]\\
&\quad = \mathbb{E}^{\mathbb{P}^*}[\overline{W}_t - \overline{W}_s/Z_{1-s}] = \mathbb{E}^{\mathbb{P}^*}[\overline{W}_t - \overline{W}_s/\bar{Z}_s].
\end{aligned}$$

To show this equals 0, it suffices to prove that, for every bounded Borelian function g with compact support included in $\mathbb{R}_+$, we have

$$\forall\, 0\leqslant s\leqslant t\leqslant 1,\quad \mathbb{E}^{\mathbb{P}^*}[g(\bar{Z}_s)(\overline{W}_t - \overline{W}_s)] = 0,$$

which is equivalent to

$$\forall\, 0\leqslant s\leqslant t\leqslant 1,\quad \mathbb{E}^{\mathbb{P}^*}\left[g(Z_t)(W_t - W_s + l_t(Z) - l_s(Z) + \int_s^t \frac{\frac{\partial}{\partial z}p(u,Z_u)}{p(u,Z_u)}\mathrm{d}u)\right] = 0, \tag{2.8}$$

by definition of $(\overline{W}_u)_{0\leqslant u\leqslant 1}$.

In order to prove (2.8), we follow the method used by Pardoux (1984/85, p. 51). We fix $t\in\,]0,1]$, and we let s vary in $[0,t]$. For every $u\in[0,t]$ and $z\in\mathbb{R}_+$, we define

$$v(u,z) = \mathbb{E}^{\mathbb{P}^*}[g(Z_t)/Z_u = z]. \tag{2.9}$$

Then, $v \in \mathscr{C}_b^{1,2}([0,t[\times\mathbb{R}_+;\mathbb{R})$ and satisfies

$$\begin{aligned}&\frac{\partial}{\partial z}v(u,0)=0,\\&\frac{\partial}{\partial u}v(u,z)+\frac{1}{2}\frac{\partial^2}{\partial z^2}v(u,z)=0.\end{aligned}\tag{2.10}$$

This is a consequence of Kolmogorov's equation satisfied by the function $h(s,z)$, and by the density $p(s,z)$ (see (2.2) and (2.3)):

$$\frac{\partial}{\partial s}p(s,z)=\frac{1}{2}\frac{\partial^2}{\partial z^2}p(s,z).$$

Then, we apply Itô's formula to the semi-martingale $v(u,Z_u)$ between times s and t:

$$\begin{aligned}g(Z_t)&=v(t,Z_t)\\&=v(s,Z_s)+\int_s^t\frac{\partial}{\partial u}v(u,Z_u)\,\mathrm{d}u+\int_s^t\frac{\partial}{\partial z}v(u,Z_u)(\mathrm{d}W_u+\tfrac{1}{2}\mathrm{d}l_u(Z))\\&\quad+\frac{1}{2}\int_s^t\frac{\partial^2}{\partial z^2}v(u,Z_u)\,\mathrm{d}u\end{aligned}\tag{2.11}$$

$$g(Z_t)=v(s,Z_s)+\int_s^t\frac{\partial}{\partial z}v(u,Z_u)\,\mathrm{d}W_u.$$

To show (2.11), we have used (2.10), and we remind the measure $dl_u(Z)$ is supported by the zeros of Z. We deduce

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^*}[g(Z_t)(W_t-W_s)]\\&=\mathbb{E}^{\mathbb{P}^*}[v(s,Z_s)(W_t-W_s)]+\mathbb{E}^{\mathbb{P}^*}\left[\int_s^t\frac{\partial}{\partial z}v(u,Z_u)\,\mathrm{d}u\right]\\&=0+\mathbb{E}^{\mathbb{P}^*}\left[\int_s^t\frac{\partial}{\partial z}v(u,Z_u)\,\mathrm{d}u\right]\quad\text{because } W \text{ is a } \mathscr{F}^Z\text{-martingale under } \mathbb{P}^*,\\&=\int_s^t\mathrm{d}u\int_0^\infty\frac{\partial}{\partial z}v(u,z)p(u,z)\,\mathrm{d}z\quad\text{then, using an integration by parts,}\\&=\int_s^t\mathrm{d}u\left\{[v(u,z)p(u,z)]_0^\infty-\int_0^\infty\frac{\partial}{\partial z}p(u,z)v(u,z)\,\mathrm{d}z\right\}\\&\qquad(v\text{ is bounded because } g \text{ too is})\\&=-\int_s^t v(u,0)p(u,0)\,\mathrm{d}u-\int_s^t\mathbb{E}^{\mathbb{P}^*}\left[v(u,Z_u)\frac{\frac{\partial}{\partial z}p(u,Z_u)}{p(u,Z_u)}\right]\mathrm{d}u\\&=-\int_s^t v(u,0)p(u,0)\,\mathrm{d}u-\int_s^t\mathbb{E}^{\mathbb{P}^*}\left[g(Z_t)\frac{\frac{\partial}{\partial z}p(u,Z_u)}{p(u,Z_u)}\right]\mathrm{d}u.\end{aligned}$$

Finally

$$\mathbb{E}^{\mathbb{P}^*}\left[g(Z_t)\left(W_t - W_s + \int_s^t \frac{\frac{\partial}{\partial z}p(u,Z_u)}{p(u,Z_u)}\,\mathrm{d}u\right)\right] = -\int_s^t v(u,0)p(u,0)\,\mathrm{d}u. \tag{2.12}$$

To prove (2.8), we now identify the right member of (2.12):

$$\mathbb{E}^{\mathbb{P}^*}[\mathrm{d}l_u(Z)] = 2\mathrm{d}\mathbb{E}^{\mathbb{P}^*}[Z_u] = 2\int_0^\infty z\frac{\partial}{\partial u}p(u,z)\,\mathrm{d}z\,\mathrm{d}u$$

$$= \int \mathrm{d}p_0(z_0)\int_0^\infty z\left\{\frac{\partial^2}{\partial z^2}h(u,z-z_0) + \frac{\partial^2}{\partial z^2}h(u,z+z_0)\right\}\mathrm{d}z\,\mathrm{d}u$$

(function h satisfies Kolmogorov's equations too)

$$= -\int \mathrm{d}p_0(z_0)\int_0^\infty \left(\frac{\partial}{\partial z}h(u,z-z_0) + \frac{\partial}{\partial z}h(u,z+z_0)\right)\mathrm{d}z\,\mathrm{d}u$$

by integration by parts

$$= \int \mathrm{d}p_0(z_0)(h(u,-z_0) + h(u,z_0))\,\mathrm{d}u = p(u,0)\,\mathrm{d}u.$$

Then

$$\int_s^t v(u,0)p(u,0)\,\mathrm{d}u = \mathbb{E}^{\mathbb{P}^*}\left[\int_s^t v(u,0)\,\mathrm{d}l_u(Z)\right]$$

$$= \mathbb{E}^{\mathbb{P}^*}\left[\int_s^t v(u,Z_u)\,\mathrm{d}l_u(Z)\right]$$

($\mathrm{d}l_u(Z)$ is supported by the zeros of $Z_.$)

$$= \mathbb{E}^{\mathbb{P}^*}[g(Z_t)(l_t(Z) - l_s(Z))]$$

applying the following lemma with $F = g$. Lemma (2.1) is then proved. □

Lemma 2.2. *Let F be a bounded Borelian function. Then*

$$\mathbb{E}^{\mathbb{P}^*}[F(Z_t)(l_t(Z) - l_s(Z))] = \mathbb{E}^{\mathbb{P}^*}\left[\int_s^t \mathbb{E}^{\mathbb{P}^*}[F(Z_t)/Z_u]\,\mathrm{d}l_u(Z)\right].$$

Proof. It is an immediate consequence of the fact that, in an integration with respect to an optional process, we can replace the integrated process by its optional projection. □

Let $\Omega \stackrel{\text{def}}{=} \mathscr{C}([0,1];\mathbb{R}_+)$ be the canonical space of continuous positive trajectories during the time interval $[0,1]$, endowed with the probability $\mathbb{P}$. We denote by $(z_t)_{0\leqslant t\leqslant 1}$ the canonical process, and by $(\mathscr{G}_t)_{0\leqslant t\leqslant 1}$ the filtration it generates. Then, $(z_t)_{0\leqslant t\leqslant 1}$ is a semi-martingale of bracket t, the local time (at level 0) of which may be defined by

$$l_t(z) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t 1_{(0\leqslant z_s\leqslant\varepsilon)}\,\mathrm{d}s$$

so that M_t defined by

$$M_t \stackrel{\text{def}}{=} z_t - z_0 - \tfrac{1}{2} l_t(z), \tag{2.13}$$

is a $\mathscr{G}_t$ Brownian motion.

We denote by $\overline{\mathbb{P}} = \mathbb{P}^* \circ \overline{Z}^{-1}$, the law of the reversed process $\overline{Z}$, so that $\overline{\mathbb{P}} = \mathbb{P} \circ R$. Lemma 2.1 implies that

$$\overline{M}_t \stackrel{\text{def}}{=} z_t - z_0 - \frac{1}{2} l_t(z) - \int_0^t \frac{\frac{\partial}{\partial z} p(1-s, z_s)}{p(1-s, z_s)} \mathrm{d}s, \tag{2.14}$$

is, under $\overline{\mathbb{P}}$, a $\mathscr{G}_t$ martingale of bracket t, i.e. a $\mathscr{G}_t$ Brownian motion.

Let us consider now a probability measure $\mathbb{Q}$, absolutely continuous with respect to $\mathbb{P}$, with finite relative entropy: $H(\mathbb{Q}|\mathbb{P}) < \infty$. Define $\overline{\mathbb{Q}} = \mathbb{Q} \circ R$. Since relative entropy does not increase under any measurable transformation (Föllmer, 1984), it is invariant under the time reverse transformation, $H(\overline{\mathbb{Q}}|\overline{\mathbb{P}}) = H(\mathbb{Q}|\mathbb{P}) < \infty$. Thanks to the entropic conditions, there exists $\mathscr{G}_t$ predictable processes $(\beta_t)_{0\leqslant t\leqslant 1}$ and $(\overline{\beta}_t)_{0\leqslant t\leqslant 1}$ such that, with obvious notations (Cattiaux and Léonard, 1994; Jacod, 1979):

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t} = \frac{\mathrm{d}q_0}{\mathrm{d}p_0} \exp\left(\int_0^t \beta_s \,\mathrm{d}M_s - \frac{1}{2}\int_0^t \beta_s^2 \,\mathrm{d}s\right),$$
$$p_0 \stackrel{\text{def}}{=} \mathbb{P} \circ z_0^{-1}, \quad q_0 \stackrel{\text{def}}{=} \mathbb{Q} \circ z_0^{-1}, \tag{2.15a}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \beta_s^2 \,\mathrm{d}s\right] < \infty, \tag{2.15b}$$

$$\left.\frac{\mathrm{d}\overline{\mathbb{Q}}}{\mathrm{d}\overline{\mathbb{P}}}\right|_{\mathscr{G}_t} = \frac{\mathrm{d}q_1}{\mathrm{d}p_1} \exp\left(\int_0^t \overline{\beta}_s \,\mathrm{d}\overline{M}_s - \frac{1}{2}\int_0^t \overline{\beta}_s^2 \,\mathrm{d}s\right),$$
$$p_1 \stackrel{\text{def}}{=} \mathbb{P} \circ z_1^{-1}, \quad q_1 \stackrel{\text{def}}{=} \mathbb{Q} \circ z_1^{-1}, \tag{2.15c}$$

$$\mathbb{E}^{\overline{\mathbb{Q}}}\left[\int_0^1 \overline{\beta}_s^2 \,\mathrm{d}s\right] < \infty, \tag{2.15d}$$

$$\Gamma_t \stackrel{\text{def}}{=} M_t - \int_0^t \beta_s \,\mathrm{d}s \text{ is a } \mathscr{G}_t \text{ Brownian motion under } \mathbb{Q}, \tag{2.16a}$$

$$\overline{\Gamma}_t \stackrel{\text{def}}{=} \overline{M}_t - \int_0^t \overline{\beta}_s \,\mathrm{d}s \text{ is a } \mathscr{G}_t \text{ Brownian motion under } \overline{\mathbb{Q}}. \tag{2.16b}$$

Proposition 2.3. *Let $f \in \mathscr{C}_K^\infty(\mathbb{R}_+^*)$. Then, for any $t \in]0,1]$*

$$\mathbb{E}^{\mathbb{Q}}[f'(z_t)] = -\,\mathbb{E}^{\mathbb{Q}}\left[f(z_t)\left(\beta_t + \overline{\beta}_{1-t} \circ R + \frac{\frac{\partial}{\partial z} p(t, z_t)}{p(t, z_t)}\right)\right].$$

The rest of Section 2.1 is dedicated to the proof of Proposition 2.3. For this, we need to prove the following:

Lemma 2.4. *For every* $\varepsilon \in]0,1]$,

$$\mathbb{E}^{\mathbb{Q}}\left[\int_\varepsilon^1 \left|\frac{\frac{\partial}{\partial z}p(t,z_t)}{p(t,z_t)}\right| \mathrm{d}t\right] < \infty.$$

Consequence. Actually, suppose Lemma 2.4 is proved. We then deduce that for almost all $t>0$, we have

$$\begin{aligned}|\mathbb{E}^{\mathbb{Q}}[f'(z_t)]| &\leqslant \|f\|_\infty \left\{\mathbb{E}^{\mathbb{Q}}[\beta_t^2]^{1/2} + \mathbb{E}^{\overline{\mathbb{Q}}}[\overline{\beta}_{1-t}^2]^{1/2} + \mathbb{E}^{\mathbb{Q}}\left[\left|\frac{\frac{\partial}{\partial z}p(t,z_t)}{p(t,z_t)}\right|\right]\right\}\\ &\leqslant \|f\|_\infty c_t.\end{aligned}$$

Thanks to Bitcheler–Jacod (1981/82), this inequality implies the existence of a density on $]0,+\infty[$, denoted by $q_t(z)$, for the law of z_t under $\mathbb{Q}$ (for almost all $t\leqslant 1$); this density is almost everywhere differentiable, and satisfies

$$q_t'(z) = q_t(z)\left\{\frac{\frac{\partial}{\partial z}p(t,z)}{p(t,z)} + \mathbb{E}^{\mathbb{Q}}[\beta_t + (\overline{\beta}_{1-t}\circ R)/z_t = z]\right\}. \tag{2.17}$$

As $\mathbb{Q}(z_t=0)=0$, (since $\mathbb{Q}\ll\mathbb{P}$ and $\mathbb{P}(z_t=0)=0$), and, because of (2.17), $q_t'(z)=0$ as soon as $q_t(z)=0$, the quantity $q_t'(z)/q_t(z)$ may be extended to $\mathbb{R}_+$ with

$$\frac{\frac{\partial}{\partial z}p(t,z)}{p(t,z)} + \mathbb{E}^{\mathbb{Q}}[\beta_t + (\overline{\beta}_{1-t}\circ R)/z_t = z].$$

We then immediately deduce:

Proposition 2.5. *For almost all* $t\in]0,1]$, *the law of* z_t *under* $\mathbb{Q}$ *has a density with respect to* $\mathbb{P}$; *this density, denoted by* $\rho_t(z) = q_t(z)/p(t,z)$, *satisfies the duality equation*:

$$\frac{\rho_t'(z)}{\rho_t(z)} \overset{\text{a.e.}}{=} \mathbb{E}^{\mathbb{Q}}[\beta_t + (\overline{\beta}_{1-t}\circ R)/z_t = z].$$

In order to prove Proposition 2.3, we use the method developed by Föllmer (1984) in the Brownian case; we write in another way each member of the following equality:

$$\mathbb{E}^{\mathbb{Q}}[f(z_t)(z_t - z_{t-h})] = -\,\mathbb{E}^{\overline{\mathbb{Q}}}[f(z_{1-t})(z_{1-t+h} - z_{1-t})] \tag{2.18}$$

where $0<h<t\leqslant 1$.

We develop the right member of (2.18) using (2.14) and (2.16). We obtain

$$\begin{aligned}&-\mathbb{E}^{\overline{\mathbb{Q}}}[f(z_{1-t})(z_{1-t+h} - z_{1-t})]\\ &= -\mathbb{E}^{\overline{\mathbb{Q}}}\left[f(z_{1-t})\int_{1-t}^{1-t+h}\left(\overline{\beta}_u + \frac{\frac{\partial}{\partial z}p(1-u,z_u)}{p(1-u,z_u)}\right)\mathrm{d}u\right]\\ &\quad -\mathbb{E}^{\overline{\mathbb{Q}}}\left[f(z_{1-t})\frac{l_{1-t+h}(z) - l_{1-t}(z)}{2}\right].\end{aligned} \tag{2.19}$$

Now we write the left member of (2.18) applying Itô's formula to the semi-martingale $f(z_s)$ between times $(t-h)$ and t. Thanks to (2.13) and (2.16), we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[f(z_t)(z_t - z_{t-h})]\\
&= \mathbb{E}^{\mathbb{Q}}\left[f(z_{t-h})\left(\Gamma_t - \Gamma_{t-h} + \int_{t-h}^{t} \beta_s\,\mathrm{d}s + \frac{l_t(z) - l_{t-h}(z)}{2}\right)\right]\\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\left(\int_{t-h}^{t} f'(z_s)(\mathrm{d}\Gamma_s + \beta_s\,\mathrm{d}s) + \frac{f'(0)}{2}\frac{l_t(z) - l_{t-h}(z)}{2} + \frac{1}{2}\int_{t-h}^{t} f''(z_s)\,\mathrm{d}s\right)\right.\\
&\qquad \left.\times\left(\Gamma_t - \Gamma_{t-h} + \int_{t-h}^{t} \beta_s\,\mathrm{d}s + \frac{l_t(z) - l_{t-h}(z)}{2}\right)\right] \qquad (2.20)\\
&= 0 + \mathbb{E}^{\mathbb{Q}}\left[f(z_{t-h})\left(\int_{t-h}^{t} \beta_s\,\mathrm{d}s + \frac{l_t(z) - l_{t-h}(z)}{2}\right)\right] + \mathbb{E}^{\mathbb{Q}}\left[\int_{t-h}^{t} f'(z_s)\,\mathrm{d}s\right]\\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\left(\int_{t-h}^{t} f'(z_s)\,\mathrm{d}\Gamma_s\right)\left(\int_{t-h}^{t} \beta_s\,\mathrm{d}s + \frac{l_t(z) - l_{t-h}(z)}{2}\right)\right]\\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\left(\int_{t-h}^{t} (f'(z_s)\beta_s + \tfrac{1}{2}f''(z_s))\,\mathrm{d}s\right)\right.\\
&\qquad \left.\times\left(\Gamma_t - \Gamma_{t-h} + \int_{t-h}^{t} \beta_s\,\mathrm{d}s + \frac{l_t(z) - l_{t-h}(z)}{2}\right)\right]
\end{aligned}
$$

(We remind f has a compact support in $]0,+\infty[$, which implies $f'(0)=0$.)
We inject (2.19) and (2.20) in (2.18) and obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[f(z_{t-h})\frac{1}{h}\int_{t-h}^{t} \beta_s \mathrm{d}s\right] + \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}\int_{t-h}^{t} f'(z_s)\,\mathrm{d}s\right]\\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[f(z_t)\frac{1}{h}\int_{t-h}^{t}\left(\overline{\beta}_{1-u}\circ R + \frac{\frac{\partial}{\partial z}p(u,z_u)}{p(u,z_u)}\right)\mathrm{d}u\right]\\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}\int_{t-h}^{t} \beta_s\,\mathrm{d}s\int_{t-h}^{t} (f'(z_s)\beta_s + \tfrac{1}{2}f''(z_s))\,\mathrm{d}s\right]\\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}\int_{t-h}^{t} \beta_s\,\mathrm{d}s\int_{t-h}^{t} f'(z_s)\,\mathrm{d}\Gamma_s\right]\\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}\int_{t-h}^{t} (f'(z_s)\beta_s + \tfrac{1}{2}f''(z_s))\,\mathrm{d}s\,(\Gamma_t - \Gamma_{t-h})\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{l_t(z) - l_{t-h}(z)}{2h}(f(z_t) - f(z_{t-h}) - \int_{t-h}^{t} f'(z_s)\,\mathrm{d}\Gamma_s\right.\\
&\qquad \left. - \int_{t-h}^{t} (f'(z_s)\beta_s + \tfrac{1}{2}f''(z_s))\,\mathrm{d}s)\right]. \qquad (2.21)
\end{aligned}
$$

We now remind a result proved by Föllmer (1984, Proposition 2.5, p. 121), and which will be very useful for our work:

Lemma 2.6. *Suppose there exists* $p\geqslant 1$ *such that* $\mathbb{E}^{\mathbb{Q}}[\int_0^1 |\eta_s|^p\,\mathrm{d}s]<\infty$. *Then, for almost all* $t\in]0,1[$,

$$\mathbb{E}^{\mathbb{Q}}\left[\int_{t-h}^{t} |\eta_s|^p\,\mathrm{d}s\right] = \mathrm{O}(h),$$

and,

$$\eta_t = \lim_{\substack{\alpha\searrow 0\\ \beta\searrow 0}} \frac{1}{\alpha+\beta}\int_{t-\alpha}^{t+\beta} \eta_s\,\mathrm{d}s$$

where the last limit is both in $\mathbb{L}^p(\mathbb{Q})$ *and* $\mathbb{Q}$ *almost surely.*

Now, we show

Lemma 2.7. *Let* $f\in\mathscr{C}_K^\infty(\mathbb{R}_+^*)$. *As* h *goes to* 0, *the left member of* (2.21) *converges to*

$$\mathbb{E}^{\mathbb{Q}}\left[f(z_t)\left(\beta_t+\overline{\beta}_{1-t}\circ R+\frac{\frac{\partial}{\partial z}p(t,z_t)}{p(t,z_t)}\right)+f'(z_t)\right].$$

Proof. As f and its derivatives are bounded, we have, thanks to (2.15b)–(2.15d), and to Lemma 2.6,

$$\frac{1}{h}\int_{t-h}^{t}\beta_s\,\mathrm{d}s \xrightarrow[\mathbb{L}^2(\mathbb{Q})]{h\searrow 0} \beta_t,$$

$$\frac{1}{h}\int_{t-h}^{t}\overline{\beta}_{1-s}\circ R\,\mathrm{d}s \xrightarrow[\mathbb{L}^2(\mathbb{Q})]{h\searrow 0} \overline{\beta}_{1-t}\circ R,$$

$$\frac{1}{h}\int_{t-h}^{t}f'(z_s)\,\beta_s\,\mathrm{d}s \xrightarrow[\mathbb{L}^2(\mathbb{Q})]{h\searrow 0} f'(z_t)\beta_t,$$

$$\frac{1}{h}\int_{t-h}^{t}g(z_s)\,\mathrm{d}s \xrightarrow[\mathbb{L}^2(\mathbb{Q})]{h\searrow 0} g(z_t)\quad\text{for every bounded Borelian function } g.$$

For the terms like $\mathbb{E}^{\mathbb{Q}}[Y_h\int_{t-h}^t g(z_s)\,\mathrm{d}\Gamma_s]$, where g is a bounded function ($g=f'$ or $g=1$), and $(Y_h)_h$ is bounded in $\mathbb{L}^2(\mathbb{Q})$ because it converges ($Y_h=\frac{1}{h}\int_{t-h}^t\beta_s\,\mathrm{d}s$, or $Y_h=\frac{1}{h}\int_{t-h}^t(f'(z_s)\beta_s+\frac{1}{2}f''(z_s))\,\mathrm{d}s$), we have the following inequality:

$$\left|\mathbb{E}^{\mathbb{Q}}\left[Y_h\int_{t-h}^{t}g(z_s)\,\mathrm{d}\Gamma_s\right]\right| \leqslant \mathbb{E}^{\mathbb{Q}}[Y_h^2]^{1/2}\,\mathbb{E}^{\mathbb{Q}}\left[\left(\int_{t-h}^{t}g(z_s)\,\mathrm{d}\Gamma_s\right)^2\right]^{1/2} \leqslant c\|g\|_\infty\sqrt{h}$$

thanks to Burkhölder–Davis–Gundy's inequalities, and to the fact that Γ is a Brownian motion under $\mathbb{Q}$. So, we obtain

$$\lim_{h\searrow 0}\mathbb{E}^{\mathbb{Q}}\left[f(z_{t-h})\frac{1}{h}\int_{t-h}^{t}\beta_s\,\mathrm{d}s\right] = \mathbb{E}^{\mathbb{Q}}[f(z_t)\,\beta_t],$$

$$\lim_{h\searrow 0}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}\int_{t-h}^{t}f'(z_s)\,\mathrm{d}s\right] = \mathbb{E}^{\mathbb{Q}}[f'(z_t)],$$

$$\lim_{h\searrow 0} \mathbb{E}^{\mathbb{Q}}\left[f(z_t)\frac{1}{h}\int_{t-h}^{t}(\overline{\beta}_{1-s}\circ R)\,\mathrm{d}s\right] = \mathbb{E}^{\mathbb{Q}}[f(z_t)(\overline{\beta}_{1-t}\circ R)],$$

$$\lim_{h\searrow 0} \mathbb{E}^{\mathbb{Q}}\left[\left(\int_{t-h}^{t} f'(z_s)\,\mathrm{d}\Gamma_s\right)\frac{1}{h}\int_{t-h}^{t}\beta_s\,\mathrm{d}s\right] = 0,$$

$$\lim_{h\searrow 0} \mathbb{E}^{\mathbb{Q}}\left[\int_{t-h}^{t}(f'(z_s)\beta_s + \tfrac{1}{2}f''(z_s))\,\mathrm{d}s\,\frac{1}{h}\int_{t-h}^{t}\beta_s\,\mathrm{d}s\right] = 0,$$

$$\lim_{h\searrow 0} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}\int_{t-h}^{t}(f'(z_s)\beta_s + \tfrac{1}{2}f''(z_s))\,\mathrm{d}s\,(\Gamma_t - \Gamma_{t-h})\right] = 0.$$

The result of Lemma 2.4, i.e.

$$\mathbb{E}^{\mathbb{Q}}\left[\int_{\varepsilon}^{1}\left|\frac{\frac{\partial}{\partial z}p(u,z_u)}{p(u,z_u)}\right|\mathrm{d}u\right] < \infty$$

for every $\varepsilon > 0$, implies, thanks to Lemma 2.6, that for every $t \in]0,1]$

$$\lim_{h\searrow 0}\frac{1}{h}\int_{t-h}^{t}\frac{\frac{\partial}{\partial z}p(u,z_u)}{p(u,z_u)}\,\mathrm{d}u \overset{\mathbb{L}^1(\mathbb{Q})}{=} \frac{\frac{\partial}{\partial z}p(t,z_t)}{p(t,z_t)},$$

and, as f is bounded,

$$\lim_{h\searrow 0} \mathbb{E}^{\mathbb{Q}}\left[f(z_t)\frac{1}{h}\int_{t-h}^{t}\frac{\frac{\partial}{\partial z}p(u,z_u)}{p(u,z_u)}\,\mathrm{d}u\right] = \mathbb{E}^{\mathbb{Q}}\left[f(z_t)\frac{\frac{\partial}{\partial z}p(t,z_t)}{p(t,z_t)}\right]. \qquad \square$$

To prove Lemma 2.4, we remind the following result about the increments of a real Brownian motion (Clark, 1970):

Lemma 2.8. *Let Γ be a real Brownian motion under $\mathbb{Q}$, and*

$$v_t^{\Gamma}(h) \overset{\text{def}}{=} \sup_{\substack{0\leqslant u\leqslant v\leqslant t\\ v-u\leqslant h}} |\Gamma_u - \Gamma_v|. \tag{2.22}$$

Then

$$\forall\delta>0,\ \exists K_\delta>0,\ \forall h\in[0,1],\ \forall T>0,\ \mathbb{E}^{\mathbb{Q}}[v_T^{\Gamma}(h)^{4(1+\delta)}]\leqslant K_\delta\, T^\delta\, h^{2+\delta}.$$

This gives controls for the local time $l_t(z)$ under $\mathbb{Q}$.

Lemma 2.9. (i) $\frac{1}{2}l_t(z)\leqslant v_t^{\Gamma}(t) + \int_0^t |\beta_s|\,\mathrm{d}s$.

(ii) *There exists a constant c such that for every $t\in[0,1]$, we have $\mathbb{E}^{\mathbb{Q}}[l_t(z)]\leqslant c\sqrt{t}$, $\mathbb{E}^{\mathbb{Q}}[l_t(z)^2]\leqslant ct$.*

Proof. (i) The process $(z_t)_{t\geqslant 0}$ satisfies the following reflected equation:

$$z_t = z_0 + M_t + \frac{1}{2}l_t(z) = \left(z_0 + \Gamma_t + \int_0^t \beta_u\,\mathrm{d}u\right) + \frac{1}{2}l_t(z), \tag{2.23}$$

which implies

$$\frac{1}{2}l_t(z) = \sup_{s\leqslant t}\left(-\left(z_0 + \Gamma_s + \int_0^s \beta_u\,\mathrm{d}u\right)\vee 0\right).$$

As $z_0 \geqslant 0$, we deduce

$$\frac{1}{2} l_t(z) \leqslant \sup_{s \leqslant t} |\Gamma_s| + \int_0^t |\beta_s| \, ds \leqslant v_t^\Gamma(t) + \int_0^t |\beta_s| \, ds.$$

(ii) Fix $t \in]0,1]$. Let $(\tilde{\Gamma}_s)_{s \geqslant 0}$ be the process defined by

$$\tilde{\Gamma}_s \stackrel{\text{def}}{=} \frac{1}{\sqrt{t}} \Gamma_{st}.$$

So, $(\tilde{\Gamma}_s)_{0 \leqslant s \leqslant 1}$ is a real Brownian motion under $\mathbb{Q}$, and $v_t^\Gamma(t) = \sqrt{t}\, v_1^{\tilde{\Gamma}}(1)$. We then deduce from Lemma 2.8, and from (2.15b) that:

$$\begin{aligned}
\frac{1}{2}\mathbb{E}^{\mathbb{Q}}[l_t(z)] &\leqslant \mathbb{E}^{\mathbb{Q}}[\sqrt{t}\, v_1^{\tilde{\Gamma}}(1)] + \mathbb{E}^{\mathbb{Q}}\left[\sqrt{t}\left(\int_0^t \beta_s^2 \, ds\right)^{1/2}\right] \\
&\leqslant \sqrt{t}\left(c_0 + \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^1 \beta_s^2 \, ds\right)^{1/2}\right]\right) \stackrel{\text{def}}{=} \lambda\sqrt{t}. \\
\frac{1}{4}\mathbb{E}^{\mathbb{Q}}[l_t(z)^2] &\leqslant 2\mathbb{E}^{\mathbb{Q}}\left[v_t^\Gamma(t)^2 + \left(\int_0^t |\beta_s| \, ds\right)^2\right] \\
&\leqslant 2\left\{t\mathbb{E}^{\mathbb{Q}}[v_1^{\tilde{\Gamma}}(1)^2] + t\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \beta_s^2 \, ds\right]\right\} \stackrel{\text{def}}{=} \mu t. \qquad \square
\end{aligned}$$

Consequence. Now, we are able to prove Lemma 2.4, which ends the proof of Lemma 2.7. For every $z \geqslant 0$ and $u \in]0,1]$, we have

(i) when $(z_t)_{0 \leqslant t \leqslant 1}$ starts from the constant z_0,

$$\left|\frac{\frac{\partial}{\partial z} p(u,z_u)}{p(u,z_u)}\right| \leqslant \frac{1}{u}(z + z_0);$$

(ii) when $(z_t)_{0 \leqslant t \leqslant 1}$ starts with the initial distribution p_0 with compact support included in $[0,K]$,

$$\begin{aligned}
\frac{\frac{\partial}{\partial z} p(u,z)}{p(u,z)} &= \frac{\int dp_0(z_0)[\frac{\partial}{\partial z} h(u,z-z_0) + \frac{\partial}{\partial z} h(u,z+z_0)]}{\int dp_0(z_0)[h(u,z-z_0) + h(u,z+z_0)]} \\
&= \frac{1}{u} \frac{\int dp_0(z_0)[(z-z_0)h(u,z-z_0) - (z+z_0)h(u,z+z_0)]}{\int dp_0(z_0)[h(u,z-z_0) + h(u,z+z_0)]}
\end{aligned}$$

from where we deduce

$$\left|\frac{\frac{\partial}{\partial z} p(u,z)}{p(u,z)}\right| \leqslant \frac{1}{u}(z + K).$$

Then, in every case, and for every $\varepsilon \in]0,1]$, we have (replace K by z_0 when the initial point is a constant):

$$\mathbb{E}^{\mathbb{Q}}\left[\int_\varepsilon^1 \left|\frac{\frac{\partial}{\partial z} p(u,z_u)}{p(u,z_u)}\right| du\right] \leqslant \mathbb{E}^{\mathbb{Q}}\left[\int_\varepsilon^1 \frac{1}{u}(z_u + K)\, du\right]$$

$$= \int_{\varepsilon}^{1} \mathbb{E}^{\mathbb{Q}}\left[K + z_0 + \Gamma_u + \frac{1}{2} l_u(z) + \int_0^u \beta_s \, \mathrm{d}s\right] \frac{\mathrm{d}u}{u} \quad \text{from (2.13) and (2.16a)}$$

$$\leqslant \int_0^1 \left\{0 + \frac{1}{2} c\sqrt{u} + \mathbb{E}^{\mathbb{Q}}\left[\sqrt{u}\left(\int_0^u \beta_s^2 \, \mathrm{d}s\right)^{1/2}\right]\right\} \frac{\mathrm{d}u}{u} - 2K \operatorname{Log} \varepsilon \quad \text{by Lemma 2.9}$$

$$\leqslant \int_0^1 \left\{\frac{1}{2} c + \mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \beta_s^2 \, \mathrm{d}s\right]^{1/2}\right\} \frac{\mathrm{d}u}{\sqrt{u}} - 2K \operatorname{Log} \varepsilon \stackrel{\text{def}}{=} c(\varepsilon) < \infty.$$

So, Lemma 2.4 is proved. □

From now on, we dedicate this work to the study of the convergence of the right member of 2.21, for which we prove the limit is 0. For this, we prove

Lemma 2.10. *Let* $g \in \mathscr{C}_K^{\infty}(\mathbb{R}_+^*)$.

(i) *If* $(Y_h)_{h \geqslant 0}$ *is a process which converges in* $\mathbb{L}^2(\mathbb{Q})$:

$$\lim_{h \searrow 0} \mathbb{E}^{\mathbb{Q}}[Y_h(l_t(z) - l_{t-h}(z))] = 0.$$

(ii) $\lim_{h \searrow 0} \mathbb{E}^{\mathbb{Q}}[g(z_t)\frac{1}{h}(l_t(z) - l_{t-h}(z))] = \lim_{h \searrow 0} \mathbb{E}^{\mathbb{Q}}[g(z_{t-h})(1/h)(l_t(z) - l_{t-h}(z))] = 0.$

(iii) $\lim_{h \searrow 0} \mathbb{E}^{\mathbb{Q}}[(1/h)(l_t(z) - l_{t-h}(z)) \int_{t-h}^t g(z_s) \, \mathrm{d}\Gamma_s] = 0.$

Consequence. Suppose Lemma 2.10 is proved. To finish the proof of Proposition 2.3, we apply Lemma 2.10(i) with the term

$$Y_h = \frac{1}{h} \int_{t-h}^{t} (f'(z_s)\beta_s + \tfrac{1}{2} f''(z_s)) \, \mathrm{d}s,$$

which converges in $\mathbb{L}^2(\mathbb{Q})$ thanks to Lemma 2.6. Then, we apply Lemma 2.10(ii)–(iii) first with f, and then with f'; these convergences, added to the ones obtained at Lemma 2.7, allow us to conclude.

Proof. Proof of Lemma 2.10(i). First, we notice that, thanks to the reflected Eq. (2.23) satisfied by $z_.$, we have

$$0 \leqslant l_t(z) - l_{t-h}(z) \leqslant 2\, v_t^M(h)$$

see (2.22) for the definition of $v_t^M(h)$

$$\leqslant 2\left(v_t^{\Gamma}(h) + \sup_{\substack{0 \leqslant u \leqslant v \leqslant t \\ v-u \leqslant h}} \int_u^v |\beta_s| \, \mathrm{d}s\right), \tag{2.24}$$

$$0 \leqslant l_t(z) - l_{t-h}(z) \leqslant 2\left(v_t^{\Gamma}(h) + \sqrt{h}\left(\int_0^1 \beta_s^2 \, \mathrm{d}s\right)^{1/2}\right). \tag{2.25}$$

Applying Cauchy–Schwarz's inequality:

$$|\mathbb{E}^{\mathbb{Q}}[(l_t(z) - l_{t-h}(z)) Y_h]| \leqslant \mathbb{E}^{\mathbb{Q}}[Y_h^2]^{1/2}\, \mathbb{E}^{\mathbb{Q}}[(l_t(z) - l_{t-h}(z))^2]^{1/2}.$$

As Y_h converges in $\mathbb{L}^2(\mathbb{Q})$, the expectations $\mathbb{E}^{\mathbb{Q}}[Y_h^2]$ are bounded. So

$$
\begin{aligned}
|\mathbb{E}^{\mathbb{Q}}[(l_t(z)-l_{t-h}(z))\,Y_h]| &\leqslant c\mathbb{E}^{\mathbb{Q}}[v_t^{\Gamma}(h)^2 + h\int_0^1 \beta_s^2\,\mathrm{d}s]^{1/2} \quad \text{because of (2.25)}\\
&\leqslant C(\mathbb{E}^{\mathbb{Q}}[v_t^{\Gamma}(h)^8]^{1/8}+\sqrt{h}) \quad \text{with Hölder's inequality and (2.15b)}\\
&\leqslant K\{(th^3)^{1/8}+\sqrt{h}\} \xrightarrow{h\searrow 0} 0 \quad \text{thanks to Lemma 2.8 with } \delta=1. \qquad \square
\end{aligned}
$$

To prove Lemma 2.10(ii) and (iii), we first suppose β is bounded by a constant C. Then, we can use the following inequalities for every $p\geqslant 1$:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathscr{G}_t}\right)^p\right] &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(p\int_0^t \beta_s\,\mathrm{d}M_s - \frac{p^2}{2}\int_0^t \beta_s^2\,\mathrm{d}s\right)\right.\\
&\qquad\left.\times\exp\left(\frac{p^2-p}{2}\int_0^t \beta_s^2\,\mathrm{d}s\right)\right]\\
&\leqslant \exp\left(\frac{p^2-p}{2}C^2\right) \stackrel{\text{def}}{=} K_p. \qquad (2.26)
\end{aligned}
$$

Proof. Proof of Lemma 2.10(ii) and (iii) when β is bounded. We fix $\alpha>0$ such that the support of g is included in $[\alpha,\infty[$, which is possible because g has a compact support in $]0,+\infty[$.

(ii)

$$
\begin{aligned}
&\left|\mathbb{E}^{\mathbb{Q}}\left[g(z_{t-h})\frac{l_t(z)-l_{t-h}(z)}{h}\right]\right|\\
&= \left|\mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathscr{G}_t} g(z_{t-h})1_{(z_{t-h}\geqslant\alpha)}\frac{l_t(z)-l_{t-h}(z)}{h}\right]\right|\\
&\leqslant \|g\|_\infty \mathbb{E}^{\mathbb{P}}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathscr{G}_t}\right)^2\right]^{1/2}\frac{1}{h}\mathbb{E}^{\mathbb{P}}[1_{(z_{t-h}\geqslant\alpha)}(l_t(z)-l_{t-h}(z))^2]^{1/2}\\
&\leqslant \|g\|_\infty\sqrt{K_2}\,\frac{1}{h}\mathbb{E}^{\mathbb{P}}[1_{(z_{t-h}\geqslant\alpha)}(l_t(z)-l_{t-h}(z))^2]^{1/2} \quad \text{because of (2.26).}
\end{aligned}
$$

But, on the event $\{z_{t-h}\geqslant\alpha,\ l_t(z)-l_{t-h}(z)>0\}$, there exists $s\in[t-h,t]$ such that $z_s=0$. In this case, using (2.23) and (2.24), we have

$$2\,v_t^M(h)\geqslant v_t^z(h)\geqslant |z_s-z_{t-h}|=z_{t-h}\geqslant\alpha.$$

We deduce

$$
\begin{aligned}
&\left|\mathbb{E}^{\mathbb{Q}}\left[g(z_{t-h})\frac{l_t(z)-l_{t-h}(z)}{h}\right]\right|\\
&\leqslant c\,\frac{1}{h}\mathbb{E}^{\mathbb{P}}[1_{(z_{t-h}\geqslant\alpha)}\,1_{(v_t^M(h)\geqslant\alpha/2)}\,(l_t(z)-l_{t-h}(z))^2]^{1/2}
\end{aligned}
$$

$$\leqslant C\,\frac{2}{h}\mathbb{E}^{\mathbb{P}}\left[\left(\frac{2\,v_t^M(h)}{\alpha}\right)^{2n}(v_t^M(h))^2\right]^{1/2} \qquad \text{because of (2.24)}, \ (\forall n \geqslant 1)$$

$$\leqslant C\left(\frac{2}{\alpha}\right)^n\frac{2}{h}\,\mathbb{E}^{\mathbb{P}}[(v_t^M(h))^{4(n+1)}]^{1/4} \quad \text{using Cauchy–Schwarz's inequality}$$

$$\leqslant K(t)\,\alpha^{-n}\,\frac{1}{h}\,h^{(2+n)/4},$$

using Lemma 2.8, with $\delta = n$, because M is a real Brownian motion under $\mathbb{P}$.

$$\left|\mathbb{E}^{\mathbb{Q}}\left[g(z_{t-h})\frac{l_t(z)-l_{t-h}(z)}{h}\right]\right| \leqslant K(t)\,\alpha^{-n}\,h^{(n-2)/4} \xrightarrow{h\searrow} 0 \quad \text{as soon as } n>2.$$

We treat the convergence of

$$\mathbb{E}^{\mathbb{Q}}\left[g(z_t)\,\frac{l_t(z)-l_{t-h}(z)}{h}\right]$$

towards 0 in the same way, after we notice

$$\{z_t \geqslant \alpha, l_t(z)-l_{t-h}(z)>0\} \subset \left\{z_t\geqslant\alpha,\ l_t(z)-l_{t-h}(z)>0,\ v_t^M(h)\geqslant\frac{\alpha}{2}\right\}.$$

(iii) In the same way, we have $l_t(z)-l_{t-h}(z)=0$ on the event $\{z_s\geqslant\alpha,\ v_t^z(h)<\alpha\}$, as soon as $s\in[t-h,t]$. Consequently,

$$\left|\mathbb{E}^{\mathbb{Q}}\left[\frac{l_t(z)-l_{t-h}(z)}{h}\int_{t-h}^t g(z_s)\,\mathrm{d}\Gamma_s\right]\right|$$

$$=\frac{1}{h}\left|\mathbb{E}^{\mathbb{Q}}\left[\int_{t-h}^t g(z_s)\,1_{(z_s\geqslant\alpha)}\,(l_t(z)-l_{t-h}(z))\,1_{(v_t^z(h)\geqslant\alpha)}\,\mathrm{d}\Gamma_s\right]\right| \stackrel{\text{def}}{=} \mathsf{A}.$$

Applying Cauchy–Schwarz's inequality

$$\mathsf{A}\leqslant\frac{1}{h}\mathbb{E}^{\mathbb{Q}}\left[\left(\int_{t-h}^t g(z_s)\,1_{(z_s\geqslant\alpha)}\,\mathrm{d}\Gamma_s\right)^2\right]^{1/2}\mathbb{E}^{\mathbb{Q}}[(l_t(z)-l_{t-h}(z))^2\,1_{(v_t^z(h)\geqslant\alpha)}]^{1/2}$$

$$\leqslant\frac{1}{h}\,\|g\|_\infty\,\sqrt{h}\,\mathbb{E}^{\mathbb{P}}\left[\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t}(l_t(z)-l_{t-h}(z))^2\,1_{(v_t^M(h)\geqslant\alpha/2)}\right]^{1/2} \stackrel{\text{def}}{=} \mathsf{B}.$$

We apply Cauchy–Schwarz's inequality again and inequality (2.24)

$$\mathsf{B}\leqslant\frac{2}{\sqrt{h}}\,\|g\|_\infty\,\mathbb{E}^{\mathbb{P}}\left[\left(\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t}\right)^2\right]^{1/4}\mathbb{E}^{\mathbb{P}}[(v_t^M(h))^4\,1_{(v_t^M(h)\geqslant\alpha/2)}]^{1/4}$$

$$\leqslant c\,K_2^{1/4}\,\|g\|_\infty\,\frac{1}{\sqrt{h}}\,\frac{2}{\alpha}\,\mathbb{E}^{\mathbb{P}}[(v_t^M(h))^8]^{1/4} \quad \text{because of 2.26.}$$

We then apply Lemma 2.8 with $\delta = 1$, and we obtain

$$\left|\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}(l_t(z) - l_{t-h}(z)) \int_{t-h}^{t} g(z_s)\,\mathrm{d}\Gamma_s\right]\right| \leqslant C(g,t)\alpha^{-1}h^{1/4} \xrightarrow{h\searrow 0} 0. \qquad \square$$

We now prove Lemma 2.10(ii) and (iii) when the drift β is not bounded. We follow the method used by Cattiaux and Léonard (1994). We first suppose $\int_0^1 \beta_s^2\,\mathrm{d}s$ is finite. We build a sequence of processes $(\beta_t^k)_{0\leqslant t\leqslant 1}$, $\mathscr{G}_t$-predictable, such that

(Hi) $|\beta_t^k| \leqslant k \wedge |\beta_t|$,

(Hii) $$Z_t^k \stackrel{\text{def}}{=} \left.\frac{\mathrm{d}\mathbb{Q}^k}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t} = \exp\left(\int_0^t \beta_s^k\,\mathrm{d}M_s - \frac{1}{2}\int_0^t (\beta_s^k)^2\,\mathrm{d}s\right) \underset{\mathbb{P}\,\text{a.s.}}{\overset{k\to\infty}{\longrightarrow}} \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t} \qquad (2.27)$$

(Hiii) Z_t^k is uniformly bounded in k in every space $\mathbb{L}^p(\mathbb{P})$.

We can write the estimates given in Lemma 2.10(ii) and (iii) with $\mathbb{Q}^k$ instead of $\mathbb{Q}$. Thanks to (Hiii), we can choose estimates which do not depend on k, but nevertheless tend to 0 with h. For example, we obtain:

$$\mathbb{E}^{\mathbb{Q}^k}\left[\frac{1}{h}(l_t(z) - l_{t-h}(z))\left|\int_{t-h}^{t} g(z_s)\,\mathrm{d}\Gamma_s\right|\right] \leqslant c\,\alpha^{-1}\,h^{1/4},$$

where c is a constant independent of k. We then apply Fatou's lemma and (Hii) to conclude:

$$\begin{aligned}
&\left|\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}(l_t(z) - l_{t-h}(z)) \int_{t-h}^{t} g(z_s)\,\mathrm{d}\Gamma_s\right]\right| \\
&\quad\leqslant \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}(l_t(z) - l_{t-h}(z))\left|\int_{t-h}^{t} g(z_s)\mathrm{d}\Gamma_s\right|\right] \\
&\quad= \mathbb{E}^{\mathbb{P}}\left[\liminf_k \left(Z_t^k\,\frac{1}{h}(l_t(z) - l_{t-h}(z))\left|\int_{t-h}^{t} g(z_s)\,\mathrm{d}\Gamma_s\right|\right)\right] \\
&\quad\leqslant \liminf_k \mathbb{E}^{\mathbb{P}}\left[Z_t^k\,\frac{1}{h}(l_t(z) - l_{t-h}(z))\left|\int_{t-h}^{t} g(z_s)\,\mathrm{d}\Gamma_s\right|\right] \\
&\quad= \liminf_k \mathbb{E}^{\mathbb{Q}^k}\left[\frac{1}{h}(l_t(z) - l_{t-h}(z))\left|\int_{t-h}^{t} g(z_s)\mathrm{d}\Gamma_s\right|\right] \leqslant c\alpha^{-1}\,h^{1/4}.
\end{aligned}$$

In the case where $\int_0^1 \beta_s^2\,\mathrm{d}s$ is not finite, we conclude as in Cattiaux and Léonard (1994) while introducing the sequence of stopping times $T_n = \inf\{t\geqslant 0;\ \int_0^t \beta_s^2\,\mathrm{d}s \geqslant n\}\wedge 1$. $\square$

2.2. *Drifts as forward and backward derivatives*

We recall $\mathscr{G}_t = \sigma(z_s;\ s\leqslant t)$. Let us define $D =]0,+\infty[$, and $\overline{\mathscr{G}}_t \stackrel{\text{def}}{=} \sigma(z_s;\ s\geqslant t)$, so that $\overline{\mathscr{G}}_t = \mathscr{G}_{1-t}\circ R$.

Proposition 2.11. *The drifts are obtained as the following limits*:

$$\lim_{h\searrow 0}\mathbb{E}^{\mathbb{Q}}\left[1_{(z_t\in D)}\frac{z_{t+h}-z_t}{h}\Big/\mathscr{G}_t\right]\overset{\text{a.s.}}{=}1_{(z_t\in D)}\ \beta_t,$$

$$\lim_{h\searrow 0}\mathbb{E}^{\mathbb{Q}}\left[1_{(z_t\in D)}\frac{z_t-z_{t-h}}{h}\Big/\overline{\mathscr{G}}_t\right]\overset{\text{a.s.}}{=}-1_{(z_t\in D)}\left(\overline{\beta}_{1-t}\circ R+\frac{\frac{\partial}{\partial z}p(1-t,z_t)}{p(1-t,z_t)}\right).$$

Proof. We first deal with the case of the forward derivative. Thanks to (2.16a), we have, since $(\Gamma_s)_{0\leqslant s\leqslant 1}$ is a $\mathscr{G}_s$ Brownian motion under $\mathbb{Q}$:

$$\mathbb{E}^{\mathbb{Q}}\left[1_{(z_t\in D)}\frac{z_{t+h}-z_t}{h}\Big/\mathscr{G}_t\right]$$

$$=1_{(z_t\in D)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{h}\int_t^{t+h}\beta_s\,\mathrm{d}s\Big/\mathscr{G}_t\right]+\mathbb{E}^{\mathbb{Q}}\left[1_{(z_t\in D)}\frac{l_{t+h}(z)-l_t(z)}{2h}\Big/\mathscr{G}_t\right].$$

The entropic hypothesis (2.15b) implies, thanks to Lemma 2.6, that

$$\lim_{h\searrow 0}\frac{1}{h}\int_t^{t+h}\beta_s\,\mathrm{d}s\underset{\text{a.s.}}{\overset{\mathbb{L}^1(\mathbb{Q})}{=}}\beta_t.$$

To study the second term, we introduce

$$Y_h=1_{(z_t\in D)}\frac{l_{t+h}(z)-l_t(z)}{2h},$$

and, for $a>0$,

$$A_a^h\overset{\text{def}}{=}\mathbb{E}^{\mathbb{Q}}\left[1_{(v_{t+1}^z(h)\geqslant a)}\frac{l_{t+h}(z)-l_t(z)}{2h}\Big/\mathscr{G}_t\right].$$

So, we want to show

$$\lim_{h\searrow 0}\mathbb{E}^{\mathbb{Q}}[Y_h/\mathscr{G}_t]\overset{\text{a.s.}}{=}0.$$

Suppose $0<h\leqslant 1$. As in the proof of Lemma 2.10, we have

$$\{l_{t+h}(z)-l_t(z)>0\}=\{l_{t+h}(z)-l_t(z)>0,\ v_{t+1}^z(h)\geqslant z_t\}.$$

Then,

$$\mathbb{E}^{\mathbb{Q}}[Y_h/\mathscr{G}_t]=1_{(z_t\in D)}A_{z_t}^h.$$

Lemma 2.12. $\forall a>0$, $\forall k>0$, $\mathbb{E}^{\mathbb{Q}}[A_a^h]=\mathrm{O}(h^k)$.

Suppose Lemma 2.12 is proved. As

$$\mathbb{Q}[A_a^{1/n}>\varepsilon]\leqslant\frac{1}{\varepsilon}\mathbb{E}^{\mathbb{Q}}[A_a^{1/n}],$$

we deduce from Borel–Cantelli Lemma and Lemma 2.12, that

$$\forall a>0,\quad \lim_{n\to\infty}A_a^{1/n}=0,\quad \mathbb{Q}\text{-almost surely},$$

since for every $\varepsilon>0$ and $a>0$, the series of general term $\mathbb{Q}[A_a^{1/n}>\varepsilon]$ is then convergent.

As the local time increases, we have the following inequalities if $h \in [\frac{1}{n+1}, \frac{1}{n}[$:

$$0 \leqslant A_a^h \leqslant \frac{n+1}{n} A_a^{1/n}.$$

So, we deduce

$$\forall a > 0, \quad \lim_{h \searrow 0} A_a^h = 0, \ \mathbb{Q}\text{-almost surely.} \tag{2.28}$$

Then, for each $a > 0$, there exists Ω_a such that $\mathbb{Q}(\Omega_a) = 1$ and

$$\forall \omega \in \Omega_a, \quad \lim_{h \searrow 0} A_a^h(\omega) = 0.$$

Denote $\Omega_0 = \bigcap_{a \in \mathbb{Q}_+^*} \Omega_a$. Then, $\mathbb{Q}(\Omega_0) = 1$, and

$$\forall \omega \in \Omega_0, \ \forall a \in \mathbb{Q}_+^*, \quad \lim_{h \searrow 0} A_a^h(\omega) = 0.$$

Noticing that $0 \leqslant A_b^h \leqslant A_a^h$ when $a < b$, we deduce

$$\forall \omega \in \Omega_0, \ \forall a > 0, \quad \lim_{h \searrow 0} A_a^h(\omega) = 0.$$

Let $\omega \in \Omega_0$:
 if $z_t(\omega) = 0$, $\mathbb{E}^{\mathbb{Q}}[Y_h/\mathscr{G}_t](\omega) = 1_{(z_t(\omega) \in D)} A_{z_t(\omega)}^h(\omega) = 0 \ \forall h > 0$;
 if $z_t(\omega) > 0$, then $a = z_t(\omega) > 0$, and $\mathbb{E}^{\mathbb{Q}}[Y_h/\mathscr{G}_t](\omega) = A_{z_t(\omega)}^h(\omega) \xrightarrow{h \searrow 0} 0$.
We conclude that $\lim_{h \searrow 0} \mathbb{E}^{\mathbb{Q}}[Y_h/\mathscr{G}_t] \overset{\text{a.s.}}{=} 0$.

In the case of the backward derivatives, we have, as in (2.24):

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[1_{(z_t \in D)} \frac{z_t - z_{t-h}}{h} \Big/ \mathscr{G}_t\right] &= \mathbb{E}^{\bar{\mathbb{Q}}}\left[1_{(z_{1-t} \in D)} \frac{z_{1-t} - z_{1-t+h}}{h} \Big/ \mathscr{G}_{1-t}\right] \circ R \\
&= \mathbb{E}^{\bar{\mathbb{Q}}}\left[1_{(z_{1-t} \in D)} \frac{l_{1-t}(z) - l_{1-t+h}(z)}{2h} \Big/ \mathscr{G}_{1-t}\right] \circ R \\
&\quad - 1_{(z_t \in D)} \mathbb{E}^{\bar{\mathbb{Q}}}\left[\frac{1}{h} \int_{1-t}^{1-t+h} \bar{\beta}_s \, \mathrm{d}s \Big/ \mathscr{G}_{1-t}\right] \circ R \\
&\quad - 1_{(z_t \in D)} \mathbb{E}^{\bar{\mathbb{Q}}}\left[\frac{1}{h} \int_{1-t}^{1-t+h} \frac{\frac{\partial}{\partial z} p(1-u, z_u)}{p(1-u, z_u)} \, \mathrm{d}u \Big/ \mathscr{G}_{1-t}\right] \circ R
\end{aligned}$$

thanks to (2.14) and to (2.16b). The hypothesis (2.15d) and the result of Lemma 2.4 imply, thanks to Lemma 2.6, the convergence of the last two terms of the right member of the latter equality, the first term of which tends to 0, for the same reasons as in the case of the forward derivative. □

Proof of Lemma 2.12. We use the same techniques as in the proof of Lemma 2.10. Suppose $h \leqslant 1$ and $p \geqslant 1$.

$$\begin{aligned}
0 &\leqslant \mathbb{E}^{\mathbb{P}}\left[\left(1_{(v_{t+1}^z(h) \geqslant a)} \frac{l_{t+h}(z) - l_t(z)}{2h}\right)^p\right] \\
&\leqslant h^{-p} \, \mathbb{E}^{\mathbb{P}}[1_{(2v_{t+1}^M(h) \geqslant a)} (v_{t+1}^M(h))^p] \quad \text{see (2.23) and (2.24)}
\end{aligned}$$

$$\leqslant h^{-p}\frac{2^k}{a^k}\mathbb{E}^{\mathbb{P}}[(v_{t+1}^M(h))^{p+k}] \quad \text{for every } k>0$$
$$\leqslant h^{-p}\frac{2^k}{a^k}\mathbb{E}^{\mathbb{P}}[(v_{t+1}^M(h))^{4(p+k)}]^{1/4} \quad \text{by Hölder's inequality}$$
$$= \mathrm{O}(h^{\frac{k+1-3p}{4}}) \quad \text{because of (2.26) and Lemma 2.8.}$$

Then, to estimate

$$\mathbb{E}^{\mathbb{Q}}[A_a^h] = \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathscr{G}_t} 1_{(v_{t+1}^z(h)\geqslant a)}\frac{l_{t+h}(z)-l_t(z)}{2h}\right],$$

we use hypothesis (Hiii) in (2.27) when the drift β satisfies the condition $\int_0^1 \beta_s^2\,\mathrm{d}s$ finite, or we do as in Cattiaux and Léonard (1994), if not. □

3. Case of a reflected diffusion

We now consider the preceding problem, replacing the reflected Brownian motion Z by a reflected diffusion X, which takes its values in $\overline{\mathscr{D}}$ ($\mathscr{D}=]0,+\infty[\times\mathbb{R}^n$), and satisfies the stochastic differential equation:

$$\begin{aligned}
&X_t=(Z_t,Y_t)\in\mathbb{R}_+\times\mathbb{R}^n,\\
&Z_t=|\mathscr{B}_t-Z_0|=Z_0+B_t^0+l_t(Z_0,\mathscr{B}),\\
&Y_t=Y_0+\int_0^t\chi_0(s,X_s)\,\mathrm{d}s+\int_0^t\chi_1(X_s)\,\mathrm{d}B_s+\int_0^t V_0(Y_s)\,\mathrm{d}l_s(Z_0,\mathscr{B}),\\
&\forall t\geqslant 0,\quad \int_0^t 1_{(Z_s=0)}\,\mathrm{d}s=0.
\end{aligned} \tag{3.1}$$

where

- χ_0 and V_0 are n-dimensional vector fields, respectively, defined on $[0,1]\times\overline{\mathscr{D}}$, and $\mathbb{R}^n$;
- χ_1 is a $n\times n$-dimensional matrix field defined on $\overline{\mathscr{D}}$;
- under $\mathbb{P}^*$, $\mathscr{B}$ and $B=(B^1,\ldots,B^n)$ are two independent Brownian motions which take their values, respectively, in $\mathbb{R}$ and $\mathbb{R}^n$;
- $Z_0\in\mathbb{R}_+$;
- $B_t^0 \stackrel{\text{def}}{=} \int_0^t sgn(\mathscr{B}_s-Z_0)\,\mathrm{d}\mathscr{B}_s$.

The first component Z of X is then a reflected Brownian motion, the law of which has the density $p(t,z)$ (see (2.2)) with respect to Lebesgue measure. As in the preceding section, the local time at level 0 of the process Z satisfies:

$$l_t(Z)=2l_t(Z_0,\mathscr{B}),$$

so we can write Eq. (3.1) in the following way:

$$\begin{aligned}
&X_t=X_0+\int_0^t b(s,X_s)\,\mathrm{d}s+\int_0^t\sigma(X_s)\,\mathrm{d}\tilde{B}_s+\frac{1}{2}\int_0^t\mu(Y_s)\,\mathrm{d}l_s(Z),\\
&b(s,x)=\begin{pmatrix}0\\ \chi_0(s,x)\end{pmatrix},\qquad \sigma(x)=\begin{pmatrix}1&0\\0&\chi_1(x)\end{pmatrix},
\end{aligned} \tag{3.2}$$

$$\tilde{B}_t = \begin{pmatrix} B_t^0 \\ B_t^1 \\ \vdots \\ B_t^n \end{pmatrix}, \qquad \mu(y) = \begin{pmatrix} 1 \\ V_0(y) \end{pmatrix}.$$

We then denote $\mathbb{P} = \mathbb{P}^* \circ X^{-1}$, the law of X under $\mathbb{P}^*$. We make the following assumptions:

(H1) The law of X_t under $\mathbb{P}^*$ has a density $\mathscr{C}^\infty$-smooth denoted by $q(t;z,y)$ (or again $q(t;x)$ where $x=(z,y)\in\mathbb{R}_+\times\mathbb{R}^n$), which satisfies $\forall t$, $\lim_{|x|\to\infty} q(t;x)=0$.

(H2) The coefficients χ_0, χ_1, and V_0 are $\mathscr{C}_b^\infty$-smooth, and the matrix field χ_1 is invertible.

We now consider the canonical space of the $\mathbb{R}_+\times\mathbb{R}^n$-valued trajectories, endowed with the probability $\mathbb{P}$. We denote by $x_t=(z_t,y_t)\in\mathbb{R}_+\times\mathbb{R}^n$ the canonical process, and by $\mathscr{G}_t$ the filtration it generates. The first component z_t is then a semi-martingale (of bracket t), the local time of which may be defined by

$$l_t(z) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t 1_{(0\leqslant z_s\leqslant\varepsilon)}\,\mathrm{d}s. \tag{3.3}$$

Then

$$\gamma_t \stackrel{\text{def}}{=} z_t - z_0 - \tfrac{1}{2} l_t(z) \tag{3.4}$$

is a $\mathscr{G}_t$ Brownian motion starting at 0.

Furthermore, if, for every $1\leqslant i\leqslant n$, we denote

$$M_t^i \stackrel{\text{def}}{=} y_t^i - y_0^i - \int_0^t \chi_0^i(s,x_s)\,\mathrm{d}s - \frac{1}{2}\int_0^t V_0^i(y_s)\,\mathrm{d}l_s(z), \tag{3.5}$$

then $M_t=(M_t^1,\dots,M_t^n)$ is a $\mathscr{G}_t$ martingale of bracket

$$\langle M^i, M^j\rangle_t = \int_0^t \sum_{k=1}^n \chi_1^{ik}(x_s)\chi_1^{jk}(x_s)\,\mathrm{d}s.$$

As the matrix field χ_1 is invertible, we can apply the representation theorem (Ikeda and Watanabe, 1981, p. 84): there exists a $\mathscr{G}_t$ Brownian motion w, $\mathbb{R}^n$-valued and starting at 0, such that

$$M_t = y_t - y_0 - \int_0^t \chi_0(s,x_s)\,\mathrm{d}s - \frac{1}{2}\int_0^t V_0(y_s)\,\mathrm{d}l_s(z) = \int_0^t \chi_1(x_s)\,\mathrm{d}w_s. \tag{3.6}$$

We now consider a probability measure $\mathbb{Q}$, absolutely continuous with respect to $\mathbb{P}$, with finite relative entropy:

$$H(\mathbb{Q}|\mathbb{P}) < \infty. \tag{3.7}$$

We also suppose

(H3) $\quad \mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \left|\frac{\operatorname{div}(aq)}{q}(s,x_s)\right|\mathrm{d}s\right] < \infty \quad$ where $a \stackrel{\text{def}}{=} \sigma\sigma^*$.

Remark. Condition (H3) is automatically satisfied as soon as one of the two following conditions is valid:

(i) $$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^1\left|\frac{\operatorname{div}(aq)}{q}(s,x_s)\right|\right)\mathrm{d}s\right]<\infty;$$

(ii) $$\int_0^1\mathbb{E}^{\mathbb{P}}\left[\exp\left|\frac{\operatorname{div}(aq)}{q}(s,x_s)\right|\right]\mathrm{d}s<\infty.$$

Actually, we use the duality of the couple of Young functions (ϕ,ψ) where $\phi(t)\overset{\text{def}}{=}e^t-t-1$, and $\psi(t)\overset{\text{def}}{=}(1+t)\operatorname{Log}(1+t)-(1+t)$ (Neveu, 1972, Proposition IX-2-2, p. 196). In fact, condition (H3) is a little too strong, because the convergence of the integral at 0 is not necessary; we just need the existence of

$$\int_\varepsilon^1\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{\operatorname{div}(aq)}{q}(s,x_s)\right|\right]\mathrm{d}s$$

for every $\varepsilon>0$.

To study the reversed process $\overline{X}_t\overset{\text{def}}{=}X_{1-t}$, we introduce the filtration:

$$\mathscr{H}_t=\sigma(\overline{X}_t)\vee\sigma(\tilde{B}_1-\tilde{B}_s;\ s\geqslant 1-t) \tag{3.8}$$

so that $(\overline{X}_t)_{0\leqslant t\leqslant 1}$ is $\mathscr{H}_t$-adapted. We shall take care not to confuse the σ-algebra generated by X_t with the value of the matrix σ at the point X_t.

Lemma 3.1. *Let $(\overline{B}_t)_{0\leqslant t\leqslant 1}$ be the $\mathbb{R}\times\mathbb{R}^n$-valued process defined by*

$$\overline{B}_t\overset{\text{def}}{=}\begin{cases}\overline{B}_t^0\overset{\text{def}}{=}\overline{Z}_t-\overline{Z}_0-\dfrac{1}{2}l_t(\overline{Z})-\displaystyle\int_0^t\frac{\frac{\partial}{\partial z}q(1-s,\overline{X}_s)}{q(1-s,\overline{X}_s)}\,\mathrm{d}s\\[2ex]\overline{B}_t^i\overset{\text{def}}{=}B_{1-t}^i-B_1^i-\displaystyle\int_0^t\frac{\operatorname{div}(\chi_1^iq)(1-s,\overline{X}_s)}{q(1-s,\overline{X}_s)}\,\mathrm{d}s\quad 1\leqslant i\leqslant n\end{cases} \tag{3.9}$$

where J^i denotes the ith row of a matrix J. Then, $(\overline{B}_t)_{0\leqslant t\leqslant 1}$ is a $\mathscr{H}_t$ Brownian motion under $\mathbb{P}^*$, and the reversed process $\overline{X}$ satisfies the following differential equation:

$$\begin{aligned}\overline{Z}_t&=\overline{Z}_0+\overline{B}_t^0+\frac{1}{2}l_t(\overline{Z})+\int_0^t\frac{\frac{\partial}{\partial z}q(1-s,\overline{X}_s)}{q(1-s,\overline{X}_s)}\,\mathrm{d}s,\\ \overline{Y}_t^i&=\overline{Y}_0^i-\int_0^t\chi_0^i(1-s,\overline{X}_s)\,\mathrm{d}s+\sum_{j=1}^n\int_0^t\frac{\chi_1^{ij}(\overline{X}_s)\operatorname{div}(\chi_1^jq)(1-s,\overline{X}_s)}{q(1-s,\overline{X}_s)}\,\mathrm{d}s\\ &\quad+\int_0^1\sum_{j=1}^n\chi_1^{ij}(\overline{X}_s)\,\mathrm{d}\overline{B}_s^j-\frac{1}{2}\int_0^1V_0^i(\overline{Y}_s)\,\mathrm{d}l_s(\overline{Z}).\end{aligned} \tag{3.10}$$

Proof. (1) First, we show

$$\forall 0\leqslant s<t\leqslant 1,\quad \mathbb{E}^{\mathbb{P}^*}[\overline{B}_t-\overline{B}_s/\mathscr{H}_s]=\mathbb{E}^{\mathbb{P}^*}[\overline{B}_t-\overline{B}_s/\sigma(\overline{X}_s)]. \tag{3.11}$$

This is a consequence of the following property: if $\mathscr{F}$ and $\mathscr{G}$ are two independent σ-algebras, and if X is a random variable independent of $\mathscr{F}$, then,

$$\mathbb{E}[X/\mathscr{F}\vee\mathscr{G}]=\mathbb{E}[X/\mathscr{G}]. \tag{3.12}$$

Indeed, if F and G are two variables, respectively, measurable with respect to $\mathscr{F}$ and $\mathscr{G}$, the independence of $\mathscr{F}$ and $\mathscr{G}$ implies:

$$\mathbb{E}[F(GX)] = \mathbb{E}[F]\,\mathbb{E}[GX] = \mathbb{E}[F]\,\mathbb{E}[\mathbb{E}(GX/\mathscr{G})] = \mathbb{E}[F]\,\mathbb{E}[G\,\mathbb{E}(X/\mathscr{G})] = \mathbb{E}[FG\,\mathbb{E}(X/\mathscr{G})].$$

We introduce then $\mathscr{F} = \sigma(\tilde{B}_1 - \tilde{B}_u;\ u \geqslant 1-s)$, and $\mathscr{G} = \sigma(\overline{X}_s)$. We notice that $(\overline{B}_t - \overline{B}_s)$ may be written as:

$$\overline{B}_t - \overline{B}_s = \begin{cases} \overline{Z}_t - \overline{Z}_s - \frac{1}{2}(l_t(\overline{Z}) - l_s(\overline{Z})) - \int_s^t F(u,\overline{X}_u)\,\mathrm{d}u \\ B_{1-t} - B_{1-s} - \int_s^t H(u,\overline{X}_u)\,\mathrm{d}u \end{cases} \tag{3.13}$$

$$= \begin{cases} Z_{1-t} - Z_{1-s} - \frac{1}{2}(l_{1-s}(Z) - l_{1-t}(Z)) - \int_{1-t}^{1-s} F(1-u,X_u)\,\mathrm{d}u \\ B_{1-t} - B_{1-s} - \int_{1-t}^{1-s} H(1-u,X_u)\,\mathrm{d}u \end{cases} \tag{3.14}$$

for some measurable functions F and H. Then, $(\overline{B}_t - \overline{B}_s)$ is $\sigma(X_u;\ 1-t \leqslant u < 1-s) \vee \sigma(B_{1-t} - B_{1-s})$-measurable, and consequently independent of $\mathscr{F}$, since $1-t < 1-s$.

(2) To prove that $(\overline{B}_t)_{0\leqslant t\leqslant 1}$ (which is a $\mathscr{H}_t$-adapted process with bracket $\langle \overline{B}^i, \overline{B}^j\rangle_t = \delta^{ij}t$), is a $\mathscr{H}_t$ Brownian motion, we just need to prove, according to (3.11), that

$$\forall g \in \mathscr{C}_K^\infty(\mathbb{R}_+ \times \mathbb{R}^n),\ \forall 0 \leqslant s < t \leqslant 1, \quad \mathbb{E}^{\mathbb{P}^*}[g(\overline{X}_s)(\overline{B}_t - \overline{B}_s)] = 0,$$

or again,

$$\forall g \in \mathscr{C}_K^\infty(\mathbb{R}_+ \times \mathbb{R}^n),\ \forall 0 \leqslant s < t \leqslant 1, \quad \mathbb{E}^{\mathbb{P}^*}[g(X_t)(\overline{B}_{1-t} - \overline{B}_{1-s})] = 0. \tag{3.15}$$

As in the case of the reflected Brownian motion, we use the method developed in Pardoux (1984/85). We fix $t \in [0,1]$, and for $u \in [0,t]$:

$$v(u;z,y) \stackrel{\text{def}}{=} \mathbb{E}^{\mathbb{P}^*}[g(X_t)/Z_u = z,\ Y_u = y] \quad (z,y) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{3.16}$$

We know (Pardoux, 1978) that v satisfies Kolmogorov's equations:

$$\begin{aligned} &\frac{\partial}{\partial u} v(u;x) + Av(u;x) = 0 \quad &\text{on } \mathscr{D} \\ &Lv(u;x) = 0 &\text{on } \partial\mathscr{D} \end{aligned} \tag{3.17}$$

where A and L are the operators, respectively on $\mathscr{D}$ and $\partial\mathscr{D}$, associated to the diffusion X, solution of the system (3.1). We apply Itô's formula to the semi-martingale $v(u;X_u)$ between times s and t to have

$$\begin{aligned} g(X_t) = v(t;X_t) = v(s;X_s) &+ \int_s^t \langle \mathrm{grad}_x v(u;X_u) | \sigma(X_u)\,\mathrm{d}\tilde{B}_u\rangle \\ &+ \int_s^t \left(\frac{\partial}{\partial u} v(u;X_u) + Av(u;X_u)\right)\mathrm{d}u + \frac{1}{2}\int_s^t Lv(u;X_u)\,\mathrm{d}l_u(Z) \end{aligned}$$

$$g(X_t) = v(s;X_s) + \int_s^t \langle \mathrm{grad}_x v(u;X_u) | \sigma(X_u)\,\mathrm{d}\tilde{B}_u\rangle. \tag{3.18}$$

As $(\tilde{B}_t)_{0\leqslant t\leqslant 1}$ is a Brownian motion under $\mathbb{P}^*$,

$$\mathbb{E}^{\mathbb{P}^*}[g(X_t)(\tilde{B}_t-\tilde{B}_s)] = 0 + \mathbb{E}^{\mathbb{P}^*}\left[\int_s^t \sigma^*(X_u)\,\mathrm{grad}_x v(u;X_u)\,\mathrm{d}u\right]$$
$$= \int_s^t \mathrm{d}u \int_{\overline{\mathscr{D}}} \sigma^*(x)\,\mathrm{grad}_x v(u;x) q(u;x)\,\mathrm{d}x.$$

We make an integration by parts in the latter integral, after we have separated the components. As v is bounded (since g is), the hypothesis (H1) implies

$$\mathbb{E}^{\mathbb{P}^*}[g(X_t)(B_t^0-B_s^0)]$$
$$= -\int_s^t \mathrm{d}u \int_{\overline{\mathscr{D}}} v(u;x)\frac{\partial}{\partial z}q(u;x)\,\mathrm{d}x - \int_s^t \mathrm{d}u \int_{\mathbb{R}^n} v(u;0,y)q(u;0,y)\,\mathrm{d}y, \tag{3.19}$$
$$\mathbb{E}^{\mathbb{P}^*}[g(X_t)(B_t^i-B_s^i)] = -\int_s^t \mathrm{d}u \int_{\overline{\mathscr{D}}} v(u;x)\,\mathrm{div}(q\chi_1^i)(u;x)\,\mathrm{d}x, \quad 1\leqslant i\leqslant n,$$

where χ_1^i is the ith row of χ_1.

We now identify the last integral in (3.19). As in Lemma 2.2, we have

$$\mathbb{E}^{\mathbb{P}^*}[g(X_t)(l_t(Z)-l_s(Z))]$$
$$= \mathbb{E}^{\mathbb{P}^*}\left[\int_s^t \mathbb{E}^{\mathbb{P}^*}[g(X_t)/Z_u]\,\mathrm{d}l_u(Z)\right]$$
$$= \mathbb{E}^{\mathbb{P}^*}\left[\int_s^t \mathbb{E}^{\mathbb{P}^*}[v(u;X_u)/Z_u]\,\mathrm{d}l_u(Z)\right] \quad \text{thanks to the definition of (3.16)}$$
$$= \mathbb{E}^{\mathbb{P}^*}\left[\int_s^t \mathrm{d}l_u(Z)\int_{\mathbb{R}^n} v(u;Z_u,y)\mathbb{E}^{\mathbb{P}^*}[Y_u\in \mathrm{d}y/Z_u]\right]$$
$$= \int_s^t \mathbb{E}^{\mathbb{P}^*}\left[\mathrm{d}l_u(Z)\int_{\mathbb{R}^n} v(u;Z_u,y)\frac{q(u;Z_u,y)}{p(u,Z_u)}\,\mathrm{d}y\right]$$
$$= \int_{\mathbb{R}^n}\mathrm{d}y\int_s^t v(u;0,y)\frac{q(u;0,y)}{p(u,0)}\,\mathrm{d}\mathbb{E}^{\mathbb{P}^*}[l_u(Z)]$$

since the measure $\mathrm{d}l_u(Z)$ is supported by the zeros of Z.

We then conclude, as in Lemma 2.1, that

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathbb{E}^{\mathbb{P}^*}[l_u(Z)] = 2\mathbb{E}^{\mathbb{P}^*}[Z_u] = 2\int_0^\infty z\frac{\partial}{\partial u}p(u,z)\,\mathrm{d}z = p(u,0). \tag{3.20}$$

So, we have

$$\mathbb{E}^{\mathbb{P}^*}[g(X_t)(l_t(Z)-l_s(Z))] = \int_{\mathbb{R}^n}\mathrm{d}y\int_s^t v(u;0,y)q(u;0,y)\,\mathrm{d}u. \tag{3.21}$$

We report in (3.19), and we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^*}\left[g(X_t)\left(B_t^0 - B_s^0 + l_t(Z) - l_s(Z) + \int_s^t \frac{\frac{\partial}{\partial z}q(u;X_u)}{q(u;X_u)}\,\mathrm{d}u\right)\right] = 0,\\
&\mathbb{E}^{\mathbb{P}^*}\left[g(X_t)\left(B_t^i - B_s^i + \int_s^t \frac{\operatorname{div}(q\chi_1^i)(u;X_u)}{q(u;X_u)}\,\mathrm{d}u\right)\right] = 0, \quad 1\leqslant i\leqslant n.
\end{aligned} \tag{3.22}$$

Eq. (3.15) is now proved. Eq. (3.10) immediately follows thanks to the properties of Itô's integral. □

Consequence. The reversed process of the initial reflected diffusion satisfies the stochastic differential equation:

$$\overline{X}_t = \overline{X}_0 + \int_0^t \overline{b}(s,\overline{X}_s)\,\mathrm{d}s + \int_0^t \sigma(\overline{X}_s)\,\mathrm{d}\overline{B}_s + \frac{1}{2}\int_0^t \overline{\mu}(\overline{Y}_s)\,\mathrm{d}l_s(\overline{Z}), \tag{3.23}$$

where

$$\overline{\mu}(y) = \begin{pmatrix} 1 \\ -V_0(y) \end{pmatrix}$$

$$\overline{b}(s;x) = -b(1-s;x) + \frac{1}{q(1-s;x)}\begin{pmatrix} \frac{\partial}{\partial z}q(1-s;x) \\ \sum_{j,k}\chi_1^{1j}(x)\,\frac{\partial}{\partial y^k}(\chi_1^{jk}q)(1-s;x) \\ \vdots \\ \sum_{j,k}\chi_1^{nj}(x)\,\frac{\partial}{\partial y^k}(\chi_1^{jk}q)(1-s;x) \end{pmatrix}.$$

Remark. Cattiaux (1988) points out an equation of type (3.23) satisfied by the reversed process of the reflected diffusion; nevertheless, to go on working, we need here to make explicit the Brownian motions which appear.

As in the first section, we denote the law of the reversed process by $\overline{\mathbb{P}} = \mathbb{P}^* \circ \overline{X}^{-1} = \mathbb{P}\circ R$, where R is Föllmer's pathwise time reversal on $\mathscr{C}([0,1])$, and we consider $\overline{\mathbb{Q}} = \mathbb{Q}\circ R$. As entropy does not change after time reversal,

$$H(\overline{\mathbb{Q}}|\overline{\mathbb{P}}) = H(\mathbb{Q}\,|\,\mathbb{P}) < \infty. \tag{3.24}$$

Our work shows that, under $\overline{\mathbb{P}}$, the process $\overline{M_t}$ defined by

$$\begin{aligned}
\overline{M}_t^0 &\stackrel{\text{def}}{=} z_t - z_0 - \frac{1}{2}l_t(z) - \int_0^t \frac{\frac{\partial}{\partial z}q(1-s;x_s)}{q(1-s;x_s)}\,\mathrm{d}s\\
\overline{M}_t^i &\stackrel{\text{def}}{=} y_t^i - y_0^i + \int_0^t \chi_0^i(1-s,x_s)\,\mathrm{d}s - \int_0^t \sum_{j,k}\frac{\chi_1^{ij}(x_s)}{q(1-s;x_s)}\,\frac{\partial}{\partial y^k}(\chi_1^{jk}q)(1-s;x_s)\,\mathrm{d}s\\
&\quad + \frac{1}{2}\int_0^t V_0^i(y_s)\,\mathrm{d}l_s(z) \quad 1\leqslant i\leqslant n
\end{aligned} \tag{3.25}$$

is a $\mathscr{G}_t$ martingale with bracket

$$\langle \overline{M}^i, \overline{M}^j \rangle_t = \int_0^t \sum_k \chi_1^{ik}(x_s)\chi_1^{jk}(x_s)\,\mathrm{d}s, \quad 1 \leqslant i,\ j \leqslant n,$$

$$\langle \overline{M}^0 \rangle_t = t, \langle \overline{M}^i, \overline{M}^0 \rangle_t = 0, \quad 1 \leqslant i \leqslant n.$$

Again, as χ_1 is an invertible matrix, we can apply the representation theorem (Ikeda and Watanabe, 1981, p. 84). There exists $\overline{\mathbb{P}}$ independent Brownian motions $\overline{\Gamma} = (\overline{\gamma}, \overline{w})$, such that

$$\begin{aligned} \overline{M}_t^0 &= \overline{\gamma}_t, \\ \overline{M}_t^i &= \int_0^t \sum_{j=1}^n \chi_1^{ij}(x_s)\,\mathrm{d}\overline{\Gamma}_s^j, \quad 1 \leqslant i \leqslant n. \end{aligned} \tag{3.26}$$

Moreover, $\mathscr{F}_t^{\overline{M}} = \mathscr{F}_t^{\overline{\Gamma}}$. The entropic condition (3.7), and Eqs. (3.4), (3.6) and (3.25), imply the existence of $\mathscr{G}_t$ predictable processes denoted by $(\beta_t)_{0 \leqslant t \leqslant 1}$ and $(\overline{\beta}_t)_{0 \leqslant t \leqslant 1}$, such that (Jacod, 1979; Cattiaux and Léonard, 1994):

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{G}_t} = \frac{\mathrm{d}q_0}{\mathrm{d}p_0} \exp\left(\int_0^t \beta_s^* \sigma(x_s)\,\mathrm{d}\Gamma_s - \frac{1}{2}\int_0^t \|\beta_s^*\sigma(x_s)\|^2\,\mathrm{d}s \right), \tag{3.27a}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \|\beta_s^*\sigma(x_s)\|^2\,\mathrm{d}s \right] < \infty, \tag{3.27b}$$

$$\left.\frac{\mathrm{d}\overline{\mathbb{Q}}}{\mathrm{d}\overline{\mathbb{P}}}\right|_{\mathscr{G}_t} = \frac{\mathrm{d}q_1}{\mathrm{d}p_1} \exp\left(\int_0^t \overline{\beta}_s^* \sigma(x_s)\,\mathrm{d}\overline{\Gamma}_s - \frac{1}{2}\int_0^t \|\overline{\beta}_s^*\sigma(x_s)\|^2\,\mathrm{d}s \right), \tag{3.27c}$$

$$\mathbb{E}^{\overline{\mathbb{Q}}}\left[\int_0^1 \|\overline{\beta}_s^*\sigma(x_s)\|^2\,\mathrm{d}s \right] < \infty. \tag{3.27d}$$

We then deduce

$$\begin{aligned} W_t &\stackrel{\text{def}}{=} \Gamma_t - \int_0^t \sigma^*(x_s)\beta_s\,\mathrm{d}s \quad \text{is a } \mathscr{G}_t \text{ Brownian motion under } \mathbb{Q}, \\ \overline{W}_t &\stackrel{\text{def}}{=} \overline{\Gamma}_t - \int_0^t \sigma^*(x_s)\overline{\beta}_s\,\mathrm{d}s \quad \text{is a } \mathscr{G}_t \text{ Brownian motion under } \overline{\mathbb{Q}}. \end{aligned} \tag{3.28}$$

Proposition 3.2. *Let* $f \in \mathscr{C}_K^\infty(\mathbb{R}_+^* \times \mathbb{R}^n)$. *Then*

$$\mathbb{E}^{\mathbb{Q}}[f(x_t)\{b(t,x_t) + \overline{b}(1-t,x_t) + a(x_t)(\beta_t + \overline{\beta}_{1-t} \circ R)\}] = -\,\mathbb{E}^{\mathbb{Q}}[a(x_t)\,\mathrm{grad}\, f(x_t)]. \tag{3.29}$$

Consequence. If we suppose the inverse of the matrix χ_1 is bounded with first derivatives bounded, we deduce from equality (3.29) and hypothesis (H3) (which imply, for almost every $t>0$, the existence of a constant K_t which is greater than

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\frac{\operatorname{div}(aq)}{q}(t,x_t)\right|\right],$$

that the law of x_t has, under $\mathbb{Q}$, a density on $]0,+\infty[\times\mathbb{R}^n$. Let us show indeed, that under this hypothesis, we have an integration by parts formula for almost all t:

$$\forall i,\ \forall f\in\mathscr{C}_K^\infty(\mathbb{R}_+^*\times\mathbb{R}^n),\quad \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial x_i}f(x_t)\right]=\mathbb{E}^{\mathbb{Q}}[f(x_t)\,U_t^i],\quad \text{where } U_t^i\in\mathbb{L}^1(\mathbb{Q}), \tag{3.30}$$

from which we classically deduce (Stroock, 1981) that the law of x_t under $\mathbb{Q}$ has a Lipschitzian density on $\mathbb{R}_+^*\times\mathbb{R}^n$.

So, let $f\in\mathscr{C}_K^\infty(\mathbb{R}_+^*\times\mathbb{R}^n)$; we denote by $(\tilde a_{pq}(x))_{pq}$ the inverse matrix $a^{-1}(x)$; let us write now the equality we obtain while taking the ith component of formula (3.29), with $\tilde a_{ji}(x)f(x)$ instead of $f(x)$. We have

$$\mathbb{E}^{\mathbb{Q}}\left[\tilde a_{ji}(x_t)f(x_t)\left\{b^i(t,x_t)+\bar b^i(1-t,x_t)+\sum_q a_{iq}(x_t)(\beta_t^q+\bar\beta_{1-t}^q\circ R)\right\}\right]$$
$$=-\,\mathbb{E}^{\mathbb{Q}}\left[f(x_t)\sum_q a_{iq}(x_t)\frac{\partial}{\partial x_q}\tilde a_{ji}(x_t)\right]-\mathbb{E}^{\mathbb{Q}}\left[\tilde a_{ji}(x_t)\sum_q a_{iq}(x_t)\frac{\partial}{\partial x_q}f(x_t)\right].$$

Adding over i, we obtain, denoting by $B(t,x)\stackrel{\text{def}}{=}b(t,x)+\bar b(1-t,x)$:

$$\mathbb{E}^{\mathbb{Q}}\left[f(x_t)\left\{\sum_i\tilde a_{ji}(x_t)B^i(t,x_t)+\sum_{i,q}\tilde a_{ji}(x_t)a_{iq}(x_t)(\beta_t^q+\bar\beta_{1-t}^q\circ R)\right.\right.$$
$$\left.\left.+\sum_{i,q}a_{iq}(x_t)\frac{\partial}{\partial x_q}\tilde a_{ji}(x_t)\right\}\right]$$
$$=-\,\mathbb{E}^{\mathbb{Q}}\left[\sum_q\frac{\partial}{\partial x_q}f(x_t)\sum_i\tilde a_{ji}(x_t)a_{iq}(x_t)\right]=-\,\mathbb{E}^{\mathbb{Q}}\left[\sum_q\frac{\partial}{\partial x_q}f(x_t)\delta_{jq}\right]$$
$$=-\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial x_q}f(x_t)\right].$$

If we summarize

$$-\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial x_j}f(x_t)\right]=\mathbb{E}^{\mathbb{Q}}\left[f(x_t)\left\{(a^{-1}(x_t)[b(t,x_t)+\bar b(1-t,x_t)]\right.\right.$$
$$\left.\left.+\beta_t+\bar\beta_{1-t}\circ R)^j+\sum_{i,q}a_{iq}(x_t)\frac{\partial}{\partial x_q}\tilde a_{ji}(x_t)\right\}\right]. \tag{3.31}$$

Eq. (3.30) is then satisfied (for almost all t), according to hypothesis (H1) and (H2) about coefficients, and to the boundedness of expressions (3.27b) and (3.27d).

Proof of Proposition 3.2. Thanks to Itô's formula, we develop each term of the following equality:

$$\mathbb{E}^{\mathbb{Q}}[f(x_t)(x_t - x_{t-h})] = -\mathbb{E}^{\bar{\mathbb{Q}}}[f(x_{1-t})(x_{1-t+h} - x_{1-t})]. \tag{3.32}$$

It becomes

$$\mathbb{E}^{\mathbb{Q}}\Bigg[\Bigg\{ f(x_{t-h}) + \int_{t-h}^{t} \langle \operatorname{grad} f(x_s) \,|\, (b(s,x_s) + a(x_s)\beta_s)\,\mathrm{d}s + \sigma(x_s)\,\mathrm{d}W_s + \tfrac{1}{2}\mu(y_s)\,\mathrm{d}l_s(z)\rangle$$

$$+ \frac{1}{2}\int_{t-h}^{t} \left(\frac{\partial^2}{\partial z^2} f(x_s) + \sum_{i,j,k} \frac{\partial^2}{\partial y_i\,\partial y_j} f(x_s)\,\chi_1^{ik}(x_s)\,\chi_1^{jk}(x_s) \right) \mathrm{d}s \Bigg\}$$

$$\left\{ \int_{t-h}^{t} (b(s,x_s) + a(x_s)\beta_s)\,\mathrm{d}s + \int_{t-h}^{t} \sigma(x_s)\,\mathrm{d}W_s + \frac{1}{2}\int_{t-h}^{t} \mu(y_s)\,\mathrm{d}l_s(z) \right\} \Bigg]$$

$$+ \mathbb{E}^{\bar{\mathbb{Q}}}\left[f(x_{1-t}) \int_{1-t}^{1-t+h} (\bar{b}(s,x_s) + a(x_s)\bar{\beta}_s)\,\mathrm{d}s + \frac{1}{2}\int_{1-t}^{1-t+h} \bar{\mu}(y_s)\,\mathrm{d}l_s(z) \right] = 0. \tag{3.33}$$

In (3.33), we recognize terms of the type of those studied in Cattiaux and Petit (1995) and in the preceding section. For the ones where "no local time" appears, we adapt the proof of Proposition 1.1 in Cattiaux and Petit (1995), or in Petit (1992), thanks to hypothesis (H2), (H3), to properties (3.27b) and (3.27d), and to the fact that W and $\overline{W}$ are Brownian motions respectively under $\mathbb{Q}$ and $\bar{\mathbb{Q}}$ (see (3.28)). These terms, after we divide by h, converge when h tends to 0, to

$$\mathbb{E}^{\mathbb{Q}}[f(x_t)\{b(t,x_t) + \bar{b}(1-t,x_t) + a(x_t)(\beta_t + \bar{\beta}_{1-t} \circ R)\} + a(x_t)\operatorname{grad} f(x_t)]. \tag{3.34}$$

For the other terms, where local time appears, we adapt the method used in the preceding section, where expressions such that $(l_t(z) - l_{t-h}(z))$ are now replaced by $\int_{t-h}^{t} g(x_s)\,\mathrm{d}l_s(z)$. We prove, as in Lemma 2.10, the following lemma, first in the case where β is bounded:

Lemma 3.3. *Let $g \in \mathscr{C}_K^{\infty}(\mathbb{R}_+^* \times \mathbb{R}^n)$ with compact support in $[\alpha, +\infty[\times \mathbb{R}^n$, where $\alpha > 0$; let $f \in \mathscr{C}_b^{\infty}(\mathbb{R}_+^* \times \mathbb{R}^n)$, and $(Y_h)_{h\geqslant 0}$ a process which converges in $\mathbb{L}^2(\mathbb{Q})$ as h tends to 0. Then*

(1) $\mathbb{E}^{\mathbb{Q}}[(\int_{t-h}^{t} f(x_s)\,\mathrm{d}l_s(z))^2] = \mathrm{O}(1)$,

(2) $\mathbb{E}^{\mathbb{Q}}[(\int_{t-h}^{t} g(x_s)\,\mathrm{d}l_s(z))^2] = \mathrm{o}(h^p)\ \forall p > 0$,

(3) $\lim_{h\to 0} \mathbb{E}^{\mathbb{Q}}[Y_h \int_{t-h}^{t} g(x_s)\,\mathrm{d}l_s(z)] = 0$,

(4) $\lim_{h\to 0}(1/h)\mathbb{E}^{\mathbb{Q}}[g(x_t)\int_{t-h}^{t} \mu(y_s)\,\mathrm{d}l_s(z)] = \lim_{h\to 0}(1/h)\mathbb{E}^{\mathbb{Q}}[g(x_{t-h})\int_{t-h}^{t} \mu(y_s)\,\mathrm{d}l_s(z)] = 0$,

(5) $\lim_{h\to 0}(1/h)\mathbb{E}^{\mathbb{Q}}[\int_{t-h}^{t} g(x_s)\,\mathrm{d}l_s(z)\int_{t-h}^{t} f(x_s)\,\mathrm{d}W_s]=0$,
(6) $\lim_{h\to 0}(1/h)\mathbb{E}^{\mathbb{Q}}[\int_{t-h}^{t} g(x_s)\,\mathrm{d}l_s(z)\int_{t-h}^{t} \mu(y_s)\,\mathrm{d}l_s(z)]=0$.

Proof. We notice that on the event $\{v_t^z(h)<\alpha,\ l_t(z)>l_{t-h}(z)\}$, there exists $s_0\in[t-h,t]$ such that $z_{s_0}=0$, and

$$\forall s\in[t-h,t],\quad 0\leqslant z_s=z_s-z_{s_0}\leqslant v_t^z(h)<\alpha.$$

So

$$\begin{aligned}\left|\int_{t-h}^{t} g(x_s)\,\mathrm{d}l_s(z)\right| &= 1_{(v_t^z(h)\geqslant\alpha)}\left|\int_{t-h}^{t} g(x_s)\,\mathrm{d}l_s(z)\right|\\ &\leqslant \|g\|_\infty\, 1_{(v_t^z(h)\geqslant\alpha)}(l_t(z)-l_{t-h}(z)). \qquad (3.35)\end{aligned}$$

The end of the proof is similar to the one of Lemma 2.10. □

Consequence. Now, the terms of identity (3.33), the convergence of which (after division by h) we have not studied yet, are of one of the types described in Lemma 3.3. So, when β is bounded, Proposition 3.2 is proved. When β is not bounded, we use approximations (2.27) made in the reflected Brownian case, and we conclude in the same way.

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